6 Differential Forms and their Integration

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6.1 Differential 1-forms

With no motivation (see Appendix) we go straight to the definitions.

Definition 1 *The set of all linear maps* $\mathbb{R}^n \to \mathbb{R}$*,* $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$*, is called the* ***dual space*** *of* $\mathbb{R}^n$*.*

The following is a definition of functions whose values are functions!

Definition 2 *A* ***differential form of degree 1 on an open set*** $U \subseteq \mathbb{R}^n$*, also known as a* 1***-form****, is a function*

$$\boldsymbol{\omega} : U \to \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})\,.$$

It is usual to write $\boldsymbol{\omega}_{\mathbf{a}}$ and not $\boldsymbol{\omega}\,(\mathbf{a})$. So, if $\mathbf{a} \in U$, then $\boldsymbol{\omega}_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ is a linear function.

Example 3 *If the scalar-valued function* $f : U \to \mathbb{R}$*, is Fréchet differentiable on* U*, an open subset of* $\mathbb{R}^n$*, then the derivative gives a* 1*-form,* df*, the* ***differential of*** f*.*

Solution Given $\mathbf{a} \in U$, the Fréchet derivative $df_{\mathbf{a}}$ is not a 1-form but it is a linear function $\mathbb{R}^n \to \mathbb{R}$ and we can define the function

$$df : U \to \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})\,, \mathbf{a} \mapsto df_{\mathbf{a}},$$

which is an example of a 1-form. ■

Not all 1-forms arise from differentiating a function. Exactly which 1-forms **do** come from differentiating functions is a question studied later.

Studying first $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$, we recall a special subset of linear functions; the **projection functions** $p^i : \mathbb{R}^n \to \mathbb{R}$ which satisfy $p^i(\mathbf{x}) = x^i$, for $1 \leq i \leq n$. These are important.

Lemma 4 *The projection functions* $\{p^i : 1 \leq i \leq n\}$ *form a basis for* $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$.

Proof let $L \in \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$ be a linear function from $\mathbb{R}^n$ to $\mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. Then

$$L(\mathbf{x}) = L\left(\sum_{i=1}^{n} x^i \mathbf{e}_i\right) = \sum_{i=1}^{n} x^i L(\mathbf{e}_i) = \sum_{i=1}^{n} p^i(\mathbf{x})\, L(\mathbf{e}_i) = \left(\sum_{i=1}^{n} L(\mathbf{e}_i)\, p^i\right)(\mathbf{x})\,.$$

True for all $\mathbf{x} \in \mathbb{R}^n$ implies the equality of functions

$$L = \sum_{i=1}^{n} L(\mathbf{e}_i)\, p^i. \tag{1}$$

Thus the set of projection functions form a Spanning Set for $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$. It is easily shown (see the Appendix) that the p^i are linearly independent. Hence $\{p^i : 1 \leq i \leq n\}$ is a basis for $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$. ■

This means that $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$ is a vector space of dimension n.

Next, studying functions $U \to \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$ we noted above that Fréchet derivatives are 1-forms. Recall from Chapter 2 that if $L : \mathbb{R}^n \to \mathbb{R}$ is a **linear** function then $dL_{\mathbf{a}} = L$ for all $\mathbf{a} \in \mathbb{R}^n$. The projection functions are linear hence $dp^i_{\mathbf{a}} = p^i$ for all $\mathbf{a} \in \mathbb{R}^n$. In fact, a different notation is used,

Definition 5 *For each* $1 \leq i \leq n$ *define the* ***constant*** 1***-form*** dx^i *on* $\mathbb{R}^n$ *by*

$$dx^i(\mathbf{a}) = p^i$$

for all $\mathbf{a} \in \mathbb{R}^n$.

Aside, it has already been noted that some authors use x^i in place of p^i, which leads to $x^i(\mathbf{x}) = x^i$. It may be confusing to use x^i for both a function and a coordinate of a vector, but it would explain using $dx^i(\mathbf{a})$ in place of $dp^i_{\mathbf{a}}$. You might also have thought that we should write $dx^i_{\mathbf{a}}$ but it is accepted convention to write $dx^i(\mathbf{a})$. Don't blame the lecturer for the illogical notation. It's historic.
End of Aside.

Let $\boldsymbol{\omega} : U \to \text{Hom}\left(\mathbb{R}^n, \mathbb{R}\right)$ be a 1-form. Let $\mathbf{a} \in U$ so $\boldsymbol{\omega}_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ is a linear function. Then by (1),

$$\boldsymbol{\omega}_{\mathbf{a}} = \sum_{i=1}^{n} \boldsymbol{\omega}_{\mathbf{a}}(\mathbf{e}_i)\, p^i.$$

Yet $p^i = dx^i(\mathbf{b})$ for *any* $\mathbf{b} \in U$. So we choose $\mathbf{b} = \mathbf{a}$, in which case

$$\boldsymbol{\omega}_{\mathbf{a}} = \sum_{i=1}^{n} \boldsymbol{\omega}_{\mathbf{a}}(\mathbf{e}_i)\, dx^i(\mathbf{a})\,. \tag{2}$$

Definition 6 *Let $\boldsymbol{\omega} : U \to \text{Hom}\left(\mathbb{R}^n, \mathbb{R}\right)$ be a 1-form. Define functions $\omega_i : U \to \mathbb{R}$ by $\omega_i(\mathbf{a}) = \boldsymbol{\omega}_{\mathbf{a}}(\mathbf{e}_i)$ for $1 \leq i \leq n$.*

With this notation (2) becomes

$$\boldsymbol{\omega}_{\mathbf{a}} = \sum_{i=1}^{n} \omega_i\left(\mathbf{a}\right) dx^i(\mathbf{a}) = \left(\sum_{i=1}^{n} \omega_i dx^i\right)(\mathbf{a})\,. \tag{3}$$

True for all $\mathbf{a} \in U$ implies the equality of functions

$$\boldsymbol{\omega} = \sum_{i=1}^{n} \omega_i dx^i. \tag{4}$$

Thus every 1-form can be written as a linear combination of the constant 1-forms dx^i with coefficients functions $\omega_i : U \to \mathbb{R}$. That is $\left\{dx^i\right\}_{1\leq i\leq n}$ form a spanning set for 1-forms. Since the $\left\{p^i\right\}_{1\leq i\leq n}$ are linearly independent then so are the dx^i which thus form a basis for the 1-forms.

Example 7

$$\boldsymbol{\omega} = x^2 y dx + (\sin x)\, dy$$

is a 1-form on $\mathbb{R}^2$. Find $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t})$ for $\mathbf{a} = (2,3)^T$ and $\mathbf{t} = (-4,5)^T$.

Solution By (3), if $\boldsymbol{\omega} = \sum_{i=1}^{n} \omega_i dx^i$, then

$$\boldsymbol{\omega}_{\mathbf{a}} = \sum_{i=1}^{n} \omega_i(\mathbf{a})\, dx^i(\mathbf{a}) = \sum_{i=1}^{n} \omega_i(\mathbf{a})\, p^i.$$

In our case,

$$\omega_1(\mathbf{x}) = x^2 y \quad \text{and} \quad \omega_2(\mathbf{x}) = \sin x$$

so, since $\mathbf{a} = (2,3)^T$, we have $\omega_1(\mathbf{a}) = 12$ and $\omega_2(\mathbf{a}) = \sin 2$. Thus

$$\boldsymbol{\omega}_{\mathbf{a}} = 12dx(\mathbf{a}) + (\sin 2)\, dy(\mathbf{a}) = 12p^1 + (\sin 2)\, p^2.$$

Given $\mathbf{t} = (-4,5)^T \in \mathbb{R}^2$ then

$$\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t}) = 12p^1\left(\begin{pmatrix} -4 \\ 5 \end{pmatrix}\right) + (\sin 2)\, p^2\left(\begin{pmatrix} -4 \\ 5 \end{pmatrix}\right) = -48 + 5\sin 2.$$

■

Example 8 *If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, U an open subset, is Fréchet differentiable then df is a 1-form and*

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

Solution We have already noted in Example 3 that df is a 1-form. In the notation of (2) we have, for any $\mathbf{a} \in U$,

$$df_{\mathbf{a}} = \sum_{i=1}^n df_{\mathbf{a}}(\mathbf{e}_i)\, dx^i(\mathbf{a})\,.$$

Yet we have seen that a Fréchet derivative evaluated on a direction equals the directional derivative, so

$$df_{\mathbf{a}}(\mathbf{e}_i) = d_{\mathbf{e}_i} f(\mathbf{a}) = d_i f(\mathbf{a}) = \frac{\partial f}{\partial x^i}(\mathbf{a})\,.$$

Hence

$$df_{\mathbf{a}} = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(\mathbf{a})\, dx^i(\mathbf{a})\,.$$

True for all $\mathbf{a} \in U$ means we have an equality of functions,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i. \tag{5}$$

■

This is, perhaps, reminiscent of the chain rule but now the objects have a new interpretation in terms of 1-forms.

As an application, this gives us an alternative way of writing the Fréchet derivatives of the product and quotient functions seen in an earlier chapter.

Example 9 *The differential of the product function* $p : \mathbb{R}^2 \to \mathbb{R}, p(\mathbf{x}) = xy$ *is*

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = ydx + xdy. \tag{6}$$

The differential of the quotient function $q : \mathbb{R} \times \mathbb{R}^\dagger \to \mathbb{R}$, $q(\mathbf{x}) = x/y$ *is*

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = \frac{1}{y}dx - \frac{x}{y^2}dy.$$

We have seen that Fréchet derivatives are 1-forms; are all 1-forms the differential of some Fréchet differentiable function?

Example 10 *Show that the* 1*-form* $xydx+ydy$ *is* ***not*** *the differential of any Fréchet differentiable function.*

Solution Assume otherwise, so there exists a Fréchet differentiable function f such that

$$df = xydx + ydy \qquad \text{that is,} \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = xydx + ydy.$$

Equate coefficients of the constant 1-forms, so

$$\frac{\partial f}{\partial x} = xy \quad \text{and} \quad \frac{\partial f}{\partial y} = y. \tag{7}$$

Integrate the first to get

$$f(\mathbf{x}) = \frac{x^2 y}{2} + g(y)\,, \tag{8}$$

for some function $g(y)$. Differentiate this w.r.t. y when we get

$$\frac{\partial f}{\partial y} = \frac{x^2}{2} + g'(y)\,.$$

But from (7) we have $\partial f/\partial y = y$ so we can equate to get

$$y = \frac{x^2}{2} + g'(y)$$

Integrate w.r.t. y to get

$$\frac{y^2}{2} = \frac{x^2}{2}y + g(y) + C, \tag{9}$$

for some constant C. Combine (8) and (9) as $f(\mathbf{x}) = y^2/2 - C$. Yet this does not satisfy $\partial f/\partial x = xy$ seen in (7). Hence no such f exists. ∎

A conclusion of this is that not all 1-forms are differentials of Fréchet differentiable functions. But if they are they are given a special name,

Definition 11 *A* 1*-form* $\boldsymbol{\omega} : U \subseteq \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ *is* ***exact*** *if there exists a* $\mathcal{C}^1$*-function* $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ *such that* $\boldsymbol{\omega} = df$.

Why demand a $\mathcal{C}^1$-function and not a Fréchet differentiable function? One reason is that later we prove a result which states that, under some conditions, a given form is 'not the differential of a C^1-function'. Under Definition 11 this conclusion becomes 'the form is not exact'. Some authors demand that f is a $\mathcal{C}^\infty$-function, i.e. all derivatives of all orders exist and are continuous. Such functions are called **smooth**.

6.1.1 Second derivatives

Definition 12 *Let* $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$*, be scalar-valued and Fréchet differentiable on* U*. If* $\partial f / \partial x^i : U \to \mathbb{R}$ *has a* j*-th partial derivative, i.e. is differentiable w.r.t.* x^j*, then this is written*

$$\frac{\partial f}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right) = \frac{\partial^2 f}{\partial x^j \partial x^i} : U \to \mathbb{R}.$$

When $j = i$ *this can be written as*

$$\frac{\partial^2 f}{\left(\partial x^i\right)^2} : U \to \mathbb{R}.$$

These are all called the ***partial derivatives of*** f ***of order*** *2.*

Similarly, we can define partial derivatives ***of order*** q *for any* $q \geq 1$ *by induction on* q *(when they exist):*

$$\frac{\partial}{\partial x^{i_q}}\left(\frac{\partial^{q-1} f}{\partial x^{i_{q-1}} \ldots \partial x^{i_2} \partial x^{i_1}}\right) = \frac{\partial^q f}{\partial x^{i_q} \partial x^{i_{q-1}} \ldots \partial x^{i_2} \partial x^{i_1}} : U \to \mathbb{R}.$$

Definition 13 *If all of the* q*-th order partial derivatives of a function* $f : U \to \mathbb{R}$ *where* U *is open in* $\mathbb{R}^n$ *exist and are continuous on* U*, then we say that* f *is a* ***function of class*** $\mathcal{C}^q$*, or a* $\mathcal{C}^q$ *-function.*

This continues the earlier definition of $\mathcal{C}^1$ as the functions whose partial derivatives exist and are continuous; $\mathcal{C}^0$ is the set of continuous functions.

Definition 14 *A 1-form $\boldsymbol{\omega}$ is of class $\mathcal{C}^q$ if its components ω_i are functions of class $\mathcal{C}^q$.*

Note If $\boldsymbol{\omega} = df$ then $\omega_i = \partial f/\partial x^i$. In this case the **form $\boldsymbol{\omega}$** is of class C^q if and only if the **function** f is of class $\mathcal{C}^{q+1}$.

An important result is that if all the derivatives of order 2 of f are continuous, i.e. if f is of class $\mathcal{C}^2$, then the order of differentiation is immaterial.

Theorem 15 *If a function $f : U \to \mathbb{R}$ where U is open in $\mathbb{R}^n$ is of class $\mathcal{C}^2$ then*

$$\frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial^2 f}{\partial x^i \partial x^j}$$

for all i and j.

Proof Given $\mathbf{a} \in U$ then, since U is an open set, there exists a $\delta > 0$ such that the open ball $B_\delta(\mathbf{a})$ centred at $\mathbf{a}$ lies within U. Suppose $1 \le i, j \le n$ with $i \ne j$. Then, for $0 < |t| < \delta$, define

$$\begin{aligned} A(t) &= f(\mathbf{a} + t\mathbf{e}_j + t\mathbf{e}_i) - f(\mathbf{a} + t\mathbf{e}_j) - f(\mathbf{a} + t\mathbf{e}_i) + f(\mathbf{a}) \\ &= \theta(t) - \theta(0), \end{aligned}$$

where

$$\theta(s) = f(\mathbf{a} + s\mathbf{e}_j + t\mathbf{e}_i) - f(\mathbf{a} + s\mathbf{e}_j),$$

for s between 0 and t. (Remember, I never said t was positive!).

The function θ is continuous (since f is continuous) and differentiable. In fact,

$$\theta'(s) = \frac{\partial f}{\partial x^j}(\mathbf{a} + s\mathbf{e}_j + t\mathbf{e}_i) - \frac{\partial f}{\partial x^j}(\mathbf{a} + s\mathbf{e}_j).$$

(See the Appendix for more details on this step.) Hence by the Mean Value Theorem for real valued functions of one variable there exists s_1 between 0 and t such that $\theta(t) - \theta(0) = \theta'(s_1)(t - 0)$. Dividing through by t and we have

$$\begin{aligned} \frac{A(t)}{t} &= \frac{\theta(t) - \theta(0)}{t} = \frac{\partial f}{\partial x^j}(\mathbf{a} + s_1\mathbf{e}_j + t\mathbf{e}_i) - \frac{\partial f}{\partial x^j}(\mathbf{a} + s_1\mathbf{e}_j) \\ &= \phi(t) - \phi(0) \end{aligned}$$

where

$$\phi(s) = \frac{\partial f}{\partial x^j}(\mathbf{a} + s_1\mathbf{e}_j + s\mathbf{e}_i),$$

for s between 0 and t. The function ϕ is continuous (since $\partial f/\partial x^j$ is continuous) and differentiable. In fact,

$$\phi'(s) = \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}(\mathbf{a} + s_1\mathbf{e}_j + s\mathbf{e}_i)\right) = \frac{\partial^2 f}{\partial x^i \partial x^j}(\mathbf{a} + s_1\mathbf{e}_j + s\mathbf{e}_i)$$

Hence by the Mean Value Theorem for real valued functions of one variable there exists s_2 between 0 and t such that $\phi(t) - \phi(0) = \phi'(s_2)(t-0)$. Dividing by t again,

$$\frac{A(t)}{t^2} = \frac{\phi(t) - \phi(0)}{t} = \frac{\partial^2 f}{\partial x^i \partial x^j}(\mathbf{a} + s_1\mathbf{e}_j + s_2\mathbf{e}_i).$$

As $t \to 0$ we also have s_1 and $s_2 \to 0$ since they are both 'stuck between' 0 and t. Thus

$$\begin{aligned}
\lim_{t\to 0}\frac{A(t)}{t^2} &= \lim_{\substack{s_i\to 0\\ s_2\to 0}} \frac{\partial^2 f}{\partial x^i \partial x^j}(\mathbf{a} + s_1\mathbf{e}_j + s_2\mathbf{e}_i) \\
&= \frac{\partial^2 f}{\partial x^i \partial x^j}\left(\lim_{\substack{s_i\to 0\\ s_2\to 0}}(\mathbf{a} + s_1\mathbf{e}_j + s_2\mathbf{e}_i)\right) \\
&\qquad \text{since } \partial^2 f/\partial x^i \partial x^j \text{ is continuous,} \\
&= \frac{\partial^2 f}{\partial x^i \partial x^j}(\mathbf{a}).
\end{aligned}$$

Since we can rearrange $A(t)$ as

$$\begin{aligned}
A(t) &= f(\mathbf{a} + t\mathbf{e}_j + t\mathbf{e}_i) - f(\mathbf{a} + t\mathbf{e}_j) - f(\mathbf{a} + t\mathbf{e}_i) + f(\mathbf{a}) \\
&= f(\mathbf{a} + t\mathbf{e}_j + t\mathbf{e}_i) - f(\mathbf{a} + t\mathbf{e}_i) - f(\mathbf{a} + t\mathbf{e}_j) + f(\mathbf{a}) \\
&\qquad \text{swapping } f(\mathbf{a} + t\mathbf{e}_j) \text{ and } f(\mathbf{a} + t\mathbf{e}_i) \\
&= f(\mathbf{a} + t\mathbf{e}_i + t\mathbf{e}_j) - f(\mathbf{a} + t\mathbf{e}_i) - f(\mathbf{a} + t\mathbf{e}_j) + f(\mathbf{a}),
\end{aligned}$$

swapping $t\mathbf{e}_j$ and $t\mathbf{e}_i$ within the first term, we can reverse the roles of i and j in the above argument. The same argument then gives

$$\lim_{t\to 0}\frac{A(t)}{t^2}=\frac{\partial^2 f}{\partial x^j \partial x^i}(\mathbf{a})\,.$$

Hence result follows. ■

Remark There are examples of twice differentiable functions $f:\mathbb{R}^2\to\mathbb{R}$ for which $\partial^2 f/\partial x^i\partial x^j \neq \partial^2 f/\partial x^j \partial x^i$. See Appendix.

Our promised property of exactness is

Theorem 16 *If a 1-form* $\boldsymbol{\omega}: U\subseteq\mathbb{R}^n\to\mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ *is a form of class* $\mathcal{C}^1$ *and exact then*

$$\frac{\partial\,\omega_i}{\partial x^j}=\frac{\partial\,\omega_j}{\partial x^i}, \tag{10}$$

for all $1\le i,j\le n$.

Proof Assume $\boldsymbol{\omega}$ is a form of class $\mathcal{C}^1$ and exact. Since $\boldsymbol{\omega}$ is exact then $\boldsymbol{\omega}=df$ for a $\mathcal{C}^1$-function $f:U\to\mathbb{R}$ and $\omega_i=\partial f/\partial x^i$ for $1\le i\le n$. Yet $\boldsymbol{\omega}$ is of class C^1 which means f is a $\mathcal{C}^2$-function. Then for any $1\le j\le n$,

$$\begin{aligned}\frac{\partial\,\omega_i}{\partial x^j} &= \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)=\frac{\partial^2 f}{\partial x^j\partial x^i}=\frac{\partial^2 f}{\partial x^i\partial x^j} \qquad \text{by Theorem 15}\\ &= \frac{\partial\,\omega_j}{\partial x^i}.\end{aligned}$$

■

Definition 17 *A* 1*-form* $\boldsymbol{\omega}:U\to\mathrm{Hom}(\mathbb{R}^n,\mathbb{R})$ *is* ***closed*** *if it's components satisfy* (10).

Example 18 *The* 1*-form* $\boldsymbol{\omega}=2xydx+(x^2+y^2)\,dy$ *is closed form since*

$$\frac{\partial(2xy)}{\partial y}=2x=\frac{\partial(x^2+y^2)}{\partial x}.$$

$\blacksquare$

Theorem 16 states that for forms of class $\mathcal{C}^1$

$$\boxed{\boldsymbol{\omega} \text{ is exact} \implies \boldsymbol{\omega} \text{ is closed.}}$$

The converse is not true, there exist closed 1-forms which are not exact, so the set of 1-forms satisfying (10) *strictly* include the set of exact 1-forms. In fact it can be shown that a 1-form $\boldsymbol{\omega}$ is exact if and only it is closed **and** U is 'simply connected'. In particular, since $\mathbb{R}^n$ is simply connected we have that $\boldsymbol{\omega}$ is exact iff $\boldsymbol{\omega}$ is closed on $\mathbb{R}^n$. This is a special case of *Poincare's Lemma for differential forms.*

To sum up, in general, for forms of class $\mathcal{C}^1$

$$\boxed{\begin{array}{c} \boldsymbol{\omega} \text{ exact} \implies \boldsymbol{\omega} \text{ is closed,} \\[2ex] \boldsymbol{\omega} \text{ is closed} \not\Longrightarrow \boldsymbol{\omega} \text{ exact.} \end{array}}$$

The contrapositive of Theorem 16 is the **Test of non-exactness:**

$$\boxed{\boldsymbol{\omega} \text{ is not closed} \implies \boldsymbol{\omega} \text{ is not exact.}}$$

This could be written as

$$\boxed{\exists\, 1 \le i, j \le n : \frac{\partial \omega_i}{\partial x^j} \neq \frac{\partial \omega_j}{\partial x^i} \implies \boldsymbol{\omega} \text{ is not exact.}}$$

6.2 Line integrals

Earlier when we looked at curves I used $\boldsymbol{\gamma}$ for both the function and the image. Here I use $\boldsymbol{\gamma}$ only for the image.

Definition 19 *A* ***differentiable curve*** $\boldsymbol{\gamma}$ *in an open set* $U \subseteq \mathbb{R}^n$ *is the image of a function* $\mathbf{g} : [a, b] \to U$ *of class* $\mathcal{C}^1$*. The initial point of the curve is* $\mathbf{g}(a)$ *and the final point is* $\mathbf{g}(b)$*. A curve is said to be* ***closed*** *if* $\mathbf{g}(a) = \mathbf{g}(b)$*.*

Definition 20 *Given a differentiable curve* $\boldsymbol{\gamma}$ *in an open set* $U \subseteq \mathbb{R}^n$ *and a* 1*-form* $\boldsymbol{\omega}$ *on* U *we define the line integral of* $\boldsymbol{\omega}$ *along* $\boldsymbol{\gamma}$ *to be*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_a^b \boldsymbol{\omega}_{\mathbf{g}(t)} \big(\mathbf{g}'(t) \big) \, dt,$$

if it exists.

So, at every point $\mathbf{g}(t)$ on $\boldsymbol{\gamma}$, we evaluate the linear function $\boldsymbol{\omega}_{\mathbf{g}(t)}$ on the tangent vector $\mathbf{g}'(t)$. This gives a real number depending on the point of the curve, that is on t. We then integrate this real-valued function of t.

There can be many functions with the same image γ. The next result states that under reasonable assumptions definition 20 is independent of which function is chosen. The assumption is that if $\boldsymbol{\gamma}$ is the image of both $\mathbf{g} : [a, b] \to U$ and $\mathbf{f} : [\alpha, \beta] \to U$ then there exists a differentiable bijection $\phi : [\alpha, \beta] \to [a, b]$ with $\phi(\alpha) = a$, $\phi(\beta) = b$ such that $\mathbf{f} = \mathbf{g} \circ \phi$.

Lemma 21 *The value of the integral does not depend on the parametrization as long as the parametrisations are in the same direction along the curve.*

Proof Let $\boldsymbol{\gamma} = \operatorname{Im} f = \operatorname{Im} g$ where $\mathbf{g} : [a, b] \to U$ and $\mathbf{f} : [\alpha, \beta] \to U$ and there exists a differentiable bijection $\phi : [\alpha, \beta] \to [a, b]$ with $\phi(\alpha) = a$, $\phi(\beta) = b$ such that $\mathbf{f} = \mathbf{g} \circ \phi$.

Recall the **Chain Rule** in the particular case $\mathbb{R} \overset{\phi}{\to} \mathbb{R} \overset{\mathbf{g}}{\to} \mathbb{R}^n$ when $(\mathbf{g} \circ \phi)'(s) = \mathbf{g}'(\phi(s)) \, \phi'(s)$. Then

$$
\begin{aligned}
\int_a^b \boldsymbol{\omega}_{\mathbf{g}(t)}\left(\mathbf{g}'(t)\right) dt &= \int_\alpha^\beta \boldsymbol{\omega}_{\mathbf{g}(\phi(s))}\big(\mathbf{g}'(\phi(s))\,\big)\phi'(s)\, ds \\
&\qquad \text{using the substitution } t = \phi(s)\,, \\
&= \int_\alpha^\beta \boldsymbol{\omega}_{\mathbf{g}(\phi(s))}\big(\mathbf{g}'(\phi(s)\,\phi'(s))\,\big)ds \qquad \text{since } \boldsymbol{\omega}_{\mathbf{g}(\phi(s))} \text{ is linear,} \\
&= \int_\alpha^\beta \boldsymbol{\omega}_{(\mathbf{g}\circ\phi)(s)}\big(\,(\mathbf{g}\circ\phi)'\,(s)\,\big)ds \qquad \text{by the Chain Rule,} \\
&= \int_\alpha^\beta \boldsymbol{\omega}_{\mathbf{f}(s)}(\mathbf{f}'\,(s))\, ds
\end{aligned}
$$

since $\mathbf{f} = \mathbf{g}\circ\phi$. ■

Example 22 *Integrate the* 1*-form*

$$\boldsymbol{\omega} = x^2 y dx + y^2 x dy + z dz,$$

along $\boldsymbol{\gamma}$*, the curve in* $\mathbb{R}^3$*, parametrised by*

$$\mathbf{g}\,(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix},$$

for $0 \le t \le 4\pi$*. (So* $\boldsymbol{\gamma}$ *is part of a spiral.)*

Solution Recall that if $\boldsymbol{\omega} = \omega_1 dx + \omega_2 dy + \omega_3 dz$, then

$$\boldsymbol{\omega}_{\mathbf{a}} = \omega_1(\mathbf{a})\, dx(\mathbf{a}) + \omega_2(\mathbf{a})\, dy(\mathbf{a}) + \omega_3(\mathbf{a})\, dz(\mathbf{a}) = \omega_1(\mathbf{a})\, p^1 + \omega_2(\mathbf{a})\, p^2 + \omega_3(\mathbf{a})\, p^3.$$

In the present case, this becomes $\boldsymbol{\omega}_{\mathbf{g}(t)} = (\cos^2 t \sin t)\, p^1 + \big(\sin^2 t \cos t\big)\, p^2 + t p^3$.

Next

$$\mathbf{g}'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix},$$

so $\boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t)) = -\cos^2 t \sin^2 t + \sin^2 t \cos^2 t + t = t$.

Therefore

$$\int_\gamma \boldsymbol{\omega} = \int_0^{4\pi} t dt = 8\pi^2.$$

$\blacksquare$

Now we get a form of the Fundamental Theorem of Calculus for exact 1-forms.

Theorem 23 *Assume $\boldsymbol{\omega}$ is exact, so $\boldsymbol{\omega} = df$ for some function f of class C^1 on an open set $U \subseteq \mathbb{R}^n$. Assume $\boldsymbol{\gamma}$ is a differentiable curve in U from $\mathbf{x}_0$ to $\mathbf{x}_1$. Then*

$$\int_{\gamma} \boldsymbol{\omega} = \int_{\gamma} df = f(\mathbf{x}_1) - f(\mathbf{x}_0).$$

Note that this is independent of the choice of $\boldsymbol{\gamma}$, it depends only on the end points.

Proof Suppose that $\boldsymbol{\gamma}$ is parametrized by $\mathbf{g} : [a, b] \to U$ so that $\mathbf{g}(a) = \mathbf{x}_0$ and $\mathbf{g}(b) = \mathbf{x}_1$. Recall the **Chain Rule**, in the special case $\mathbb{R} \xrightarrow{\mathbf{g}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$, when $(f \circ \mathbf{g})'(t) = df_{\mathbf{g}(t)}\big(\mathbf{g}'(t)\big)$. Then

$$\begin{aligned}
\int_{\gamma} df &= \int_a^b df_{\mathbf{g}(t)}\big(\mathbf{g}'(t)\big)dt \quad \text{by definition} \\
&= \int_a^b (f \circ \mathbf{g})'(t)\,dt \quad \text{by the Chain Rule} \\
&= (f \circ \mathbf{g})(b) - (f \circ \mathbf{g})(a),
\end{aligned}$$

by the Fundamental Theorem of Calculus which requires $(f \circ \mathbf{g})'$ to be continuous. That f' is continuous follows since f is a function of class C^1. And $\mathbf{g}'$ is continuous by the definition of differentiable curve. Continuing,

$$\int_{\gamma} df = f(\mathbf{g}(b)) - f(\mathbf{g}(a)) = f(\mathbf{x}_1) - f(\mathbf{x}_0).$$

$\blacksquare$

Corollary 24 *Assume that $\boldsymbol{\omega} : U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ is an exact 1-form on an open set $U \subseteq \mathbb{R}^n$ and $\boldsymbol{\gamma}$ is a closed differentiable curve in U then $\int_{\gamma} \boldsymbol{\omega} = 0$.*

Proof Immediate from Theorem since $\boldsymbol{\gamma}$ closed means that $\mathbf{x}_0 = \mathbf{x}_1$. $\blacksquare$

Remark This result provides another method for showing that a 1-form in **not** exact.

$$\boxed{\gamma \text{ closed and } \int_\gamma \boldsymbol{\omega} \neq 0 \implies \boldsymbol{\omega} \text{ is not exact.}}$$

See the Problem Sheet for an application of this principle.

6.3 Differential 2-forms

Definition 25 $\text{Hom}\left(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right)$ *is the set of all maps from* $\mathbb{R}^n \times \mathbb{R}^n$ *to* $\mathbb{R}$ *which are linear in both variables (i.e. they are bilinear).*

So if $L \in \text{Hom}\left(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right)$ then, for $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$,

$$\begin{aligned} L(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) &= \alpha L(\mathbf{x}, \mathbf{z}) + \beta L(\mathbf{y}, \mathbf{z}), \\ L(\mathbf{x}, \alpha\mathbf{w} + \beta\mathbf{z}) &= \alpha L(\mathbf{x}, \mathbf{w}) + \beta L(\mathbf{x}, \mathbf{z}). \end{aligned}$$

Definition 26 *A* ***differential form of degree*** 2*, also known as a* 2***-form****, is a function* $\boldsymbol{\omega} : U \subseteq \mathbb{R}^n \to \text{Hom}\left(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right)$ *such that for all* $\mathbf{a} \in U$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *we have* $\boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{v}, \mathbf{u}\right) = -\boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{u}, \mathbf{v}\right)$*.*

See the Appendix for some motivation for this definition and in particular the requirement that $\boldsymbol{\omega}_{\mathbf{a}}$ is *anti-symmetric*, i.e. $\boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{v}, \mathbf{u}\right) = -\boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{u}, \mathbf{v}\right)$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. It rests on the idea that $\boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{u}, \mathbf{v}\right)$ represents the area between $\mathbf{u}$ and $\mathbf{v}$ and if they are swapped the area changes sign.

Note that being anti-symmetric means that $\boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{u}, \mathbf{u}\right) = 0$ for all $\mathbf{u} \in \mathbb{R}^n$.

Repeat the argument seen earlier: If $L \in \text{Hom}\left(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}\right)$ then, for $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} L(\mathbf{u}, \mathbf{v}) &= L\left(\sum_{i=1}^n u^i \mathbf{e}_i, \sum_{j=1}^n v^j \mathbf{e}_j\right) = \sum_{i=1}^n \sum_{j=1}^n u^i v^j L(\mathbf{e}_i, \mathbf{e}_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n L(\mathbf{e}_i, \mathbf{e}_j)\, p^i(\mathbf{u})\, p^j(\mathbf{u}). \end{aligned} \tag{11}$$

Let $\boldsymbol{\omega}$ be a 2-form on an open set $U \subseteq \mathbb{R}^n$ and let $\mathbf{a} \in U$, then $\boldsymbol{\omega}_{\mathbf{a}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. By (11) with $L = \boldsymbol{\omega}_{\mathbf{a}}$,

$$\begin{aligned} \boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{u}, \mathbf{v}\right) &= \sum_{i.,j=1}^n \boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{e}_i, \mathbf{e}_j\right) p^i(\mathbf{u})\, p^j(\mathbf{v}) \\ &= \sum_{\substack{i.,j=1 \\ i \neq j}}^n \boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{e}_i, \mathbf{e}_j\right) p^i(\mathbf{u})\, p^j(\mathbf{v}) \qquad \text{since } \boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{e}_i, \mathbf{e}_i\right) = 0, \\ &= \sum_{1 \leq i < j \leq n} \boldsymbol{\omega}_{\mathbf{a}}\left(\mathbf{e}_i, \mathbf{e}_j\right) \left(p^i(\mathbf{u})\, p^j(\mathbf{v}) - p^j(\mathbf{u})\, p^i(\mathbf{v})\right), \end{aligned} \tag{12}$$

by the assumption $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{u},\mathbf{v}) = -\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{v},\mathbf{u})$ for a 2-form. Here

$$p^i(\mathbf{u})\,p^j(\mathbf{v}) - p^j(\mathbf{u})\,p^i(\mathbf{v}) = u^i v^j - u^j v^i = \det\begin{pmatrix} u^i & v^i \\ u^j & v^j \end{pmatrix}.$$

Think of this as the area of the parallelogram between the projections of $\mathbf{u}$ and $\mathbf{v}$ onto the i-j-th plane. This should be motivation for the following definition.

Definition 27 *For $1 \le i,j \le n$ define $p^i \wedge p^j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by*

$$p^i \wedge p^j(\mathbf{u},\mathbf{v}) = \det\begin{pmatrix} u^i & v^i \\ u^j & v^j \end{pmatrix},$$

for all $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$. We say p^i 'wedge' p^j.

These bilinear functions are anti-symmetric $p^i \wedge p^j = -p^j \wedge p^i$ from which it follows that $p^i \wedge p^i = 0$.

Example *Calculate* $p^1 \wedge p^2(\mathbf{v}_1,\mathbf{v}_2)$, $p^1 \wedge p^3(\mathbf{v}_1,\mathbf{v}_2)$ and $p^2 \wedge p^3(\mathbf{v}_1,\mathbf{v}_2)$ for $\mathbf{v}_1 = (1,2,-1)^T$ and $\mathbf{v}_2 = (-1,0,1)^T$.

Solution

$$\begin{aligned} p^1 \wedge p^2(\mathbf{v}_1,\mathbf{v}_2) &= \det\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = 2, \\ p^1 \wedge p^3(\mathbf{v}_1,\mathbf{v}_2) &= \det\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0, \\ p^2 \wedge p^3(\mathbf{v}_1,\mathbf{v}_2) &= \det\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} = 2. \end{aligned}$$

■

Returning to (12) we can now write

$$\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{u},\mathbf{v}) = \sum_{1\le i<j\le n} \boldsymbol{\omega}_{\mathbf{a}}(\mathbf{e}_i,\mathbf{e}_j)\,p^i \wedge p^j(\mathbf{u},\mathbf{v}),$$

for all $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$. Thus

$$\boldsymbol{\omega}_{\mathbf{a}} = \sum_{1\le i<j\le n} \boldsymbol{\omega}_{\mathbf{a}}(\mathbf{e}_i,\mathbf{e}_j)\,p^i \wedge p^j, \tag{13}$$

for all $\mathbf{a} \in U$. Motivated by Definition 6 we have

Definition 28 *Let $\boldsymbol{\omega}$ be a 2-form on an open set $U \subseteq \mathbb{R}^n$. Define functions $\omega_{i,j} : U \to \mathbb{R}$ by $\omega_{i,j}(\mathbf{a}) = \boldsymbol{\omega}_{\mathbf{a}}(\mathbf{e}_i, \mathbf{e}_j)$ for $1 \le i, j \le n$.*

Just as in Definition 5 we defined dx^i to be the constant 1-form such that $dx^i(\mathbf{a}) = p^i$ for all $\mathbf{a} \in \mathbb{R}^n$, we define special constant 2-forms:

Definition 29 *Write $dx^i \wedge dx^j$ to be the constant 2-form on $\mathbb{R}^n$ such that for all $\mathbf{a} \in \mathbb{R}^n$*

$$dx^i \wedge dx^j(\mathbf{a}) = p^i \wedge p^j.$$

With these two definitions (13) can be rewritten as

$$\boldsymbol{\omega}_{\mathbf{a}} = \sum_{1 \le i < j \le n} \omega_{i,j}(\mathbf{a})\, dx^i \wedge dx^j(\mathbf{a})$$

for all $\mathbf{a} \in U$. Hence

$$\boldsymbol{\omega} = \sum_{1 \le i < j \le n} \omega_{i,j} dx^i \wedge dx^j. \tag{14}$$

Thus all 2-forms can be written as a linear combination of the $\binom{n}{2}$ terms $dx^i \wedge dx^j$, $1 \le i < j \le n$ with coefficients $\omega_{i,j} : U \to \mathbb{R}$. It can be shown that the $dx^i \wedge dx^j$, $1 \le i < j \le n$ are linearly independent (see Appendix), and hence form a basis for 2-forms..

Example 30 *The general 2-form on $U \subset \mathbb{R}^3$ has the form*

$$\omega_{1,2}\, dx \wedge dy + \omega_{1,3}\, dx \wedge dz + \omega_{2,3}\, dy \wedge dz \tag{15}$$

where each $\omega_{i,j} : U \to \mathbb{R}$.

For a particular example.

Example 31 *Let*

$$\boldsymbol{\alpha} = xz\, dx \wedge dy + (x + yz)\, dx \wedge dz + x^2 z\, dy \wedge dz$$

be a 2-form on $\mathbb{R}^3$. Evaluate $\boldsymbol{\alpha}_{\mathbf{a}}(\mathbf{v}_1, \mathbf{v}_2)$ at the point $\mathbf{a} = (1, -1, 2)^T$ with $\mathbf{v}_1 = (1, 2, -1)^T$ and $\mathbf{v}_2 = (-1, 0, 1)^T$:

Solution The function $\boldsymbol{\alpha}_{\mathbf{a}} \in \mathrm{Hom}\,(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R})$ is

$$\boldsymbol{\alpha}_{\mathbf{a}} = 2\,p^1 \wedge p^2 - p^1 \wedge p^3 + 2\,p^2 \wedge p^3.$$

Evaluated at $(\mathbf{v}_1, \mathbf{v}_2)$ it is

$$\boldsymbol{\alpha}_{\mathbf{a}}\,(\mathbf{v}_1, \mathbf{v}_2) = 2\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 4 - 0 + 4 = 8.$$

■

6.4 Surface Integrals

Definition 32 *Let S be a parameterized 2 dimensional surface in $\mathbb{R}^n$, so there exists a Fréchet differentiable $\mathbf{g} : D \to \mathbb{R}^n$ for some $D \subseteq \mathbb{R}^2$ for which*

$$S = \{\mathbf{g}(\mathbf{x}) : \mathbf{x} \in D,\ J\mathbf{g}(\mathbf{x}) \ \text{of full rank}\}.$$

Let $\boldsymbol{\omega}$ be a 2-form defined on an open set $U \subseteq \mathbb{R}^n$ containing S, i.e. $S \subseteq U$. Then the integral of $\boldsymbol{\omega}$ over S is defined to be

$$\int_S \boldsymbol{\omega} = \int\int_D \boldsymbol{\omega}_{\mathbf{g}(s,t)} \left(\frac{\partial \mathbf{g}(s,t)}{\partial s}, \frac{\partial \mathbf{g}(s,t)}{\partial t} \right) dsdt,$$

if it exists.

In attempting to understand this we know that at every point $\mathbf{g}(s,t)$ on the surface, the directional derivatives $\mathbf{v}_1 = d_1\mathbf{g}(s,t)$ and $\mathbf{v}_2 = d_2\mathbf{g}(s,t)$ are two, independent vectors in the tangent space. Then $\boldsymbol{\omega}_{\mathbf{g}(s,t)}(\mathbf{v}_1, \mathbf{v}_2)$ is a measure of the area of the parallelogram between $\mathbf{v}_1$ and $\mathbf{v}_2$ in the tangent Space. This is then integrated over the surface.

In the definition the set $D \subseteq \mathbb{R}^2$ need not be open (and if not, we can only require that $\mathbf{g}$ is Fréchet differentiable on the *interior* points of D). If D is not open the surface may have a boundary as in the next example.

Example 33 *Evaluate*

$$\int_S (x+z)\, dx \wedge dy + x^2\, dx \wedge dz + xy\, dy \wedge dz,$$

where S is the parametrized surface

$$\left\{ \begin{pmatrix} s \\ s+t \\ st \end{pmatrix} : 0 \le s \le 1,\ 0 \le t \le 1 \right\}.$$

Solution In this case $\mathbf{g}(s,t) = (s, s+t, st)^T$ and $\boldsymbol{\omega} = (x+z)\, dx \wedge dy + x^2\, dx \wedge dz + xy\, dy \wedge dz$. So

$$\boldsymbol{\omega}_{\mathbf{g}(s,t)} = (s+st)\, p^1 \wedge p^2 + s^2\, p^1 \wedge p^3 + s(s+t)\, p^2 \wedge p^3.$$

Next $d_1\mathbf{g}((s,t)^T) = (1,1,t)^T$ and $d_2\mathbf{g}((s,t)^T) = (0,1,s)^T$. Hence,

$$\begin{aligned}
&\boldsymbol{\omega}_{\mathbf{g}(s,t)}\left(\frac{\partial \mathbf{g}\,(s,t)}{\partial s}, \frac{\partial \mathbf{g}\,(s,t)}{\partial t}\right) \\
&\qquad = \left((s+st)\,p^1 \wedge p^2 + s^2\,p^1 \wedge p^3 + s(s+t)\,p^2 \wedge p^3\right)\left(\begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ s \end{pmatrix}\right) \\
&\qquad = \left((s+st)\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + s^2 \begin{vmatrix} 1 & 0 \\ t & s \end{vmatrix} + s(s+t)\begin{vmatrix} 1 & 1 \\ t & s \end{vmatrix}\right) \\
&\qquad = s + st + s^3 + s(s+t)(s-t) \\
&\qquad = s + st + s^3 + s^3 - st^2.
\end{aligned}$$

Finally

$$\begin{aligned}
&\int_S (x+z)\,dx \wedge dy + x^2\,dx \wedge dz + xy\,dy \wedge dz \\
&\qquad\qquad = \int_0^1 \int_0^1 (s + st + s^3 + s^3 - st^2)\,ds\,dt \\
&\qquad\qquad = \int_0^1 \left[\frac{s^2}{2} + \frac{s^2 t}{2} + \frac{2s^4}{4} - \frac{s^2 t^2}{2}\right]_{s=0}^{s=1} dt \\
&\qquad\qquad = \int_0^1 \left(\frac{1}{2} + \frac{t}{2} + \frac{1}{2} - \frac{t^2}{2}\right) dt \\
&\qquad\qquad = \frac{1}{2} + \frac{1}{4} + \frac{1}{2} - \frac{1}{6} = \frac{13}{12}.
\end{aligned}$$

■

6.5 Products of 1-forms

Be aware that the definition of $dx^i \wedge dx^j$ above is of the *object* and not a *binary operation* $\wedge$. Nonetheless we can define the binary operation by

Definition 34 *Given dx^i and dx^j, constant 1-forms on $U \subseteq R^n$, the **wedge product** (or **exterior product**) is $dx^i \wedge dx^j$ given by Definition 29. Extend this by linearity to the wedge product of general 1-forms.*

The wedge product of two 1-forms is, by definition, a 2-form.

Example. Let $\boldsymbol{\alpha} = x\,dx + z\,dy + xy\,dz$, and $\boldsymbol{\beta} = y\,dx - z\,dy + x\,dz$ be 1-forms on $\mathbb{R}^3$. Then

$$\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\beta} &= (x\,dx + z\,dy + xy\,dz) \wedge (y\,dx - z\,dy + x\,dz) \\
&= xy\,dx \wedge dx - xz\,dx \wedge dy + x^2\,dx \wedge dz \\
&\qquad +yz\,dy \wedge dx - z^2\,dy \wedge dy + xz\,dy \wedge dz \\
&\qquad\qquad +xy^2\,dz \wedge dx - xyz\,dz \wedge dy + x^2 y\,dz \wedge dz \\
&= -xz\,dx \wedge dy + x^2\,dx \wedge dz + yz\,dy \wedge dx \\
&\qquad +xz\,dy \wedge dz + xy^2\,dz \wedge dx - xyz\,dz \wedge dy
\end{aligned}$$

since $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$. Then using $dy \wedge dx = -dx \wedge dy$ etc. we get

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (-xz - yz)\,dx \wedge dy + (x^2 - xy^2)\,dx \wedge dz + (xz + xyz)\,dy \wedge dz.$$

■

6.6 Differentials of 1-forms.

For completeness,

Definition 35 *A* ***0-form*** *on $U \subseteq \mathbb{R}^n$ is a function $f : U \to \mathbb{R}$.*

In this terminology, the differential of a 0-form f is

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i, \tag{16}$$

a 1-form. Perhaps we can give a definition of a differential of a 1-form and it will be a 2-form.

Without motivation (for that see the Appendix) we give:

Definition 36 *Suppose that*

$$\boldsymbol{\omega} = \sum_{j=1}^{n} \omega_j dx^j$$

is a 1-form of class $\mathcal{C}^1$ on an open set $U \subseteq \mathbb{R}^n$, where $\omega_i : U \to \mathbb{R}$. The ***exterior differential*** *$d\boldsymbol{\omega}$ is the 2-form on U given by*

$$d\boldsymbol{\omega} = \sum_{j=1}^{n} d\omega_j \wedge dx^j.$$

Example 37 *Find the differential of the 1-form $\boldsymbol{\alpha} = xyz\, dx + x^2 y\, dy + (x - yz)\, dz$ on $\mathbb{R}^3$.*

Solution

$$\begin{aligned} d\boldsymbol{\alpha} &= d\,(xyz) \wedge dx + d\left(x^2 y\right) \wedge dy + d\,(x - yz) \wedge dz \\ &= (yz\, dx + xz\, dy + xy\, dz) \wedge dx + (2xy\, dx + x^2\, dy) \wedge dy + (dx - z\, dy - y\, dz) \wedge dz \\ &= xz\, dy \wedge dx + xy\, dz \wedge dx + 2xy\, dx \wedge dy + dx \wedge dz - z\, dy \wedge dz, \end{aligned}$$

since $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$. Then using $dy \wedge dx = -dx \wedge dy$ etc. we get

$$d\boldsymbol{\alpha} = (-xz + 2xy)\, dx \wedge dy + (-xy + 1)\, dx \wedge dz - z\, dy \wedge dz.$$

■

Proposition 38 *If* $\boldsymbol{\omega} = \sum_{j=1}^{n} \omega_j \, dx^j$ *is a* 1*-form of class* $\mathcal{C}^1$ *on an open set* $U \subset \mathbb{R}^n$ *then*

$$d\boldsymbol{\omega} = \sum_{1 \le i < j \le n} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j.$$

Proof By definition

$$d\boldsymbol{\omega} = \sum_{j=1}^{n} d\omega_j \wedge dx^j.$$

From (16)

$$d\omega_j = \sum_{i=1}^{n} \frac{\partial \omega_j}{\partial x^i} dx^i.$$

Thus,

$$d\boldsymbol{\omega} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial \omega_j}{\partial x^i} dx^i \right) \wedge dx^j = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \ne j}}^{n} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j, \tag{17}$$

using $dx^i \wedge dx^i = 0$. Continuing we collect together (i, j) and (j, i) terms, so

$$\begin{aligned} d\boldsymbol{\omega} &= \sum_{1 \le i < j \le n} \left(\frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i \right) \\ &= \sum_{1 \le i < j \le n} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j, \end{aligned}$$

by the anti-symmetric property $dx^j \wedge dx^i = -dx^i \wedge dx^j$. ■

Corollary 39 *A* 1*-form* $\boldsymbol{\omega}$ *of class* $\mathcal{C}^1$ *is closed if and only if* $d\boldsymbol{\omega} = 0$.

Proof This is immediate from Definition 17 which states that $\boldsymbol{\omega}$ is closed if and only if

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0,$$

for all $1 \le i < j \le n$. ■

Later we define forms of degree > 1. Then $d\boldsymbol{\omega} = 0$ becomes the definition of closed.

Corollary 40 *If f is a function of class $\mathcal{C}^2$ then $d^2 f = 0$.*

Proof If f is of class $\mathcal{C}^2$ then df is an exact 1-form of class $\mathcal{C}^1$ which, by Theorem 16, must therefore be closed. Then, by Corollary 39, $d(df) = 0$, i.e. $d^2 f = 0$. ■

These results say that if $\boldsymbol{\omega} = df$ for some f then $d\boldsymbol{\omega} = 0$. The converse, if $d\boldsymbol{\omega} = 0$ then $\boldsymbol{\omega} = df$ for some f, need not be true. The conditions under which it is true is the subject of Poincare's Lemma mentioned earlier.

6.7 Stokes' Theorem

Recall an earlier example where our surface had a boundary. In the next important result an integral over the boundary is compared with an integral over the surface. As an example let S be a surface in $\mathbb{R}^3$ with boundary ∂S. Let ω be a 1-form defined on S. Then one can integrate the 1-form ω over ∂S. Alternatively you can differentiate ω and integrate the resulting 2-form $d\omega$ over the surface S. Stoke's Theorem says that the results are the same! The result holds in great generality but we have only time for this simple version.

Suppose that $\mathbf{g} : [0,1] \times [0,1] \to \mathbb{R}^n$ is a function of class $\mathcal{C}^2$ parametrizing $S \subset \mathbb{R}^n$. Then the boundary of S, denoted ∂S, may be parametrized using four differentiable curves:

$$\begin{array}{llll}
\mathbf{g}_b\colon [0,1] \to \mathbb{R}^n & \text{given by} & \mathbf{g}_b(s) = \mathbf{g}([s,0]^T) & \text{with image } \mathcal{B}, \\
\mathbf{g}_t\colon [0,1] \to \mathbb{R}^n & \text{given by} & \mathbf{g}_t(s) = \mathbf{g}([s,1]^T) & \text{with image } \mathcal{T}, \\
\mathbf{g}_l\colon [0,1] \to \mathbb{R}^n & \text{given by} & \mathbf{g}_l(t) = \mathbf{g}([0,t]^T) & \text{with image } \mathcal{L}, \\
\mathbf{g}_r\colon [0,1] \to \mathbb{R}^n & \text{given by} & \mathbf{g}_r(t) = \mathbf{g}([1,t]^T) & \text{with image } \mathcal{R}.
\end{array}$$

Then we can prove the following result.

Theorem 41 ***Stokes' Theorem - special case*** *If $\boldsymbol{\phi}$ is a 1-form of class $\mathcal{C}^1$ on $U \subseteq \mathbb{R}^n$ and S is a surface in U parametrized by a function of class $\mathcal{C}^2$. Then*

$$\int_S d\phi = \int_{\partial S} \phi = \int_{\mathcal{R}} \phi - \int_{\mathcal{L}} \phi - \int_{\mathcal{T}} \phi + \int_{\mathcal{B}} \phi.$$

Though it might be lost in the notation, we are going around the square $[0,1] \times [0,1]$ in a counter clockwise direction, from $(0,0)$ to $(1,0)$, using $\mathbf{g}_b$; from $(1,0)$ to $(1,1)$ using $\mathbf{g}_r$; from $(1,1)$ to $(0,1)$ using $\mathbf{g}_t$ (but in the reverse direction) and from $(0,1)$ back to $(0,0)$ using $\mathbf{g}_l$ (but again in the reverse direction). You can see this reflected in the choice of names for the image lines, $\mathcal{B}$ for bottom edge of the square, $\mathcal{R}$ for the right hand edge etc.

We now calculate the value of a 2-form at a point on a surface, calculated on two tangent vectors to the surface.

Lemma 42 *Let $\mathbf{g} : V \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ be a $\mathcal{C}^2$-function. Let ϕ be a 1-form on an open set $U \subseteq \mathbb{R}^n$ containing the image of $\mathbf{g}$. Then, for $\mathbf{s} = (s,t)^T \in V$,*

$$\begin{aligned} \frac{\partial}{\partial s}\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial t}(\mathbf{s})\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial \phi_i}{\partial x^j}(\mathbf{g}(\mathbf{s}))\,\frac{\partial g^j}{\partial s}(\mathbf{s})\,\frac{\partial g^i}{\partial t}(\mathbf{s}) \\ &\quad + \sum_{i=1}^{n}\phi_i(\mathbf{g}(\mathbf{s}))\,\frac{\partial^2 g^i}{\partial s \partial t}(\mathbf{s}). \end{aligned} \tag{18}$$

Further, the 2-form $d\phi$ satisfies

$$d\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial s}(\mathbf{s}), \frac{\partial \mathbf{g}}{\partial t}(\mathbf{s})\right) = \frac{\partial}{\partial s}\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial t}(\mathbf{s})\right) - \frac{\partial}{\partial t}\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial s}(\mathbf{s})\right). \tag{19}$$

Proof We saw within the proof of Proposition 38 that the differential of $\phi = \sum_{i=1}^n \phi_i dx^i$ is

$$d\phi = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial \phi_j}{\partial x^i}dx^i \wedge dx^j,$$

So

$$d\phi_{\mathbf{g}(\mathbf{s})} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial \phi_j}{\partial x^i}(\mathbf{g}(\mathbf{s}))\,p^i \wedge p^j.$$

Thus

$$d\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial s}(\mathbf{s}), \frac{\partial \mathbf{g}}{\partial t}(\mathbf{s})\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial \phi_j}{\partial x^i}(\mathbf{g}(\mathbf{s}))\left(\frac{\partial g^j}{\partial s}(\mathbf{s})\,\frac{\partial g^i}{\partial t}(\mathbf{s}) - \frac{\partial g^j}{\partial t}(\mathbf{s})\,\frac{\partial g^i}{\partial s}(\mathbf{s})\right). \tag{20}$$

Next we examine the RHS of (19). From

$$\phi = \sum_{i=1}^{n} \phi_i dx^i,$$

we get

$$\phi_{\mathbf{g}(\mathbf{s})} = \sum_{i=1}^{n} \phi_i \left(\mathbf{g}\left(\mathbf{s}\right)\right) p^i.$$

Thus

$$\begin{aligned} \phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}\left(\mathbf{s}\right)}{\partial t}\right) &= \sum_{i=1}^{n} \phi_i \left(\mathbf{g}\left(\mathbf{s}\right)\right) p^i \left(\frac{\partial \mathbf{g}}{\partial t}\left(\mathbf{s}\right)\right) \\ &= \sum_{i=1}^{n} \phi_i \left(\mathbf{g}\left(\mathbf{s}\right)\right) \frac{\partial g^i}{\partial t}\left(\mathbf{s}\right). \end{aligned}$$

Then, by the Product Rule for differentiation,

$$\frac{\partial}{\partial s}\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial t}\left(\mathbf{s}\right)\right) = \sum_{i=1}^{n} \frac{\partial}{\partial s}\phi_i \left(\mathbf{g}\left(\mathbf{s}\right)\right) \frac{\partial g^i}{\partial t}\left(\mathbf{s}\right) + \sum_{i=1}^{n} \phi_i \left(\mathbf{g}\left(\mathbf{s}\right)\right) \frac{\partial^2 g^i}{\partial s \partial t}\left(\mathbf{s}\right).$$

Use the Chain Rule inside the first sum to give

$$\frac{\partial}{\partial s}\phi_i \left(\mathbf{g}\left(\mathbf{s}\right)\right) = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x^j}\left(\mathbf{g}\left(\mathbf{s}\right)\right) \frac{\partial g^j}{\partial s}\left(\mathbf{s}\right).$$

Combine these last two lines to get (18).

Repeat, interchanging s and t to get

$$\begin{aligned} \frac{\partial}{\partial t}\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial s}\left(\mathbf{s}\right)\right) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x^j}\left(\mathbf{g}\left(\mathbf{s}\right)\right) \frac{\partial g^j}{\partial t}\left(\mathbf{s}\right) \frac{\partial g^i}{\partial s}\left(\mathbf{s}\right) \\ &\quad + \sum_{i=1}^{n} \phi_i \left(\mathbf{g}\left(\mathbf{s}\right)\right) \frac{\partial^2 g^i}{\partial t \partial s}\left(\mathbf{s}\right). \end{aligned}$$

Subtract the two results. Since $\mathbf{g}$ is $\mathcal{C}^2$ a result in Chapter 6 is that

$$\frac{\partial^2 g^i}{\partial t \partial s}\left(\mathbf{s}\right) = \frac{\partial^2 g^i}{\partial s \partial t}\left(\mathbf{s}\right),$$

and so these terms cancel, leaving

$$\frac{\partial}{\partial s}\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial t}(\mathbf{s})\right)-\frac{\partial}{\partial t}\phi_{\mathbf{g}(\mathbf{s})}\left(\frac{\partial \mathbf{g}}{\partial s}(\mathbf{s})\right)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial \phi_i}{\partial x^j}(\mathbf{g}(\mathbf{s}))\frac{\partial g^j}{\partial s}(\mathbf{s})\frac{\partial g^i}{\partial t}(\mathbf{s})-\frac{\partial \phi_i}{\partial x^j}(\mathbf{g}(\mathbf{s}))\frac{\partial g^j}{\partial t}(\mathbf{s})\frac{\partial g^i}{\partial s}(\mathbf{s})\right)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial \phi_i}{\partial x^j}(\mathbf{g}(\mathbf{s}))\left(\frac{\partial g^j}{\partial s}(\mathbf{s})\frac{\partial g^i}{\partial t}(\mathbf{s})-\frac{\partial g^j}{\partial t}(\mathbf{s})\frac{\partial g^i}{\partial s}(\mathbf{s})\right).$$

Yet this has been seen previously in (20) and this equality gives the last claim of the lemma. ■

Proof of Stokes' Theorem By definition

$$\begin{aligned}\int_S d\phi &= \int_0^1\int_0^1 d\phi_{g(s,t)}\left(\frac{\partial \mathbf{g}}{\partial s}(s,t),\frac{\partial \mathbf{g}}{\partial t}(s,t)\right)dsdt\\ &= \int_0^1\int_0^1\left(\frac{\partial}{\partial s}\phi_{\mathbf{g}(s,t)}\left(\frac{\partial \mathbf{g}}{\partial t}(s,t)\right)-\frac{\partial}{\partial t}\phi_{\mathbf{g}(s,t)}\left(\frac{\partial \mathbf{g}}{\partial s}(s,t)\right)\right)dsdt \qquad (21)\end{aligned}$$

by Lemma 42. Consider this as a difference of two iterated integrals. The first

$$\begin{aligned}\int_0^1\left(\int_0^1\frac{\partial}{\partial s}\phi_{\mathbf{g}(s,t)}\left(\frac{\partial \mathbf{g}}{\partial t}(s,t)\right)ds\right)dt &= \int_0^1\left(\phi_{\mathbf{g}(1,t)}\left(\frac{\partial \mathbf{g}}{\partial t}(1,t)\right)-\phi_{\mathbf{g}(0,t)}\left(\frac{\partial \mathbf{g}}{\partial t}(0,t)\right)\right)dt\\ &= \int_0^1\phi_{\mathbf{g}_r(t)}\left(\frac{\partial \mathbf{g}_r}{\partial t}(t)\right)dt-\int_0^1\phi_{\mathbf{g}_l(t)}\left(\frac{\partial \mathbf{g}_l}{\partial t}(t)\right)dt\\ &= \int_{\mathcal{R}}\phi-\int_{\mathcal{L}}\phi,\end{aligned}$$

by the definition of integration of a 1-form. Similarly, the second integral in (21) is

$$\int_0^1\left(\int_0^1\frac{\partial}{\partial t}\phi_{\mathbf{g}(s,t)}\left(\frac{\partial \mathbf{g}}{\partial s}(s,t)\right)dt\right)ds=\int_{\mathcal{T}}\phi-\int_{\mathcal{B}}\phi.$$

Hence

$$\int_S d\boldsymbol{\alpha}=\left(\int_{\mathcal{R}}\phi-\int_{\mathcal{L}}\phi\right)-\left(\int_{\mathcal{T}}\phi-\int_{\mathcal{B}}\phi\right),$$

giving the required result. ■

This is a special case of Stokes' Theorem in that the surface is an image of a square. We can allow more general surfaces. A usual form of Stokes' Theorem might be

Theorem 43 *Let S be a smooth surface in $\mathbb{R}^n$, bounded by a closed curve ∂S. Assume that the surface is orientable, and that the boundary curve is oriented so that the surface lies to the left of the curve (as you walk around the boundary, standing 'up' in the direction of the orientated normal to the surface). Let $\boldsymbol{\phi}$ be a 1-form of class $\mathcal{C}^1$ defined on a subset $U \subseteq \mathbb{R}^n$ containing S. Then*

$$\int_S d\phi = \int_{\partial S} \phi.$$

A 'flat' version of this was given in MATH10121, where S lies in $\mathbb{R}^2$ (specifically the x - y plane inside $\mathbb{R}^n$).

Example In Question Sheet 9 the following questions are asked:

i. Integrate the 1 - form $\boldsymbol{\omega} = (x + y + z)\, dx + y^2 dy + xydz$ along the boundary of the unit circle in the x - y plane, centre the origin, in the counter-clockwise direction.

ii. Integrate the 2-form $\boldsymbol{\beta} = -dx \wedge dy + (y - 1)\, dx \wedge dz + xdy \wedge dz$ over

    a. the upper half of the unit sphere, so $x^2 + y^2 + z^2 = 1$ with $z \geq 0$,

    b. the region in the x - y plane $x^2 + y^2 \leq 1$.

The connection between these questions is that the differential of $\boldsymbol{\omega}$ in part i. is the $\boldsymbol{\beta}$ in part ii. That is $d\boldsymbol{\omega} = \boldsymbol{\beta}$. Also the unit circle of part i. is the boundary of the surfaces in parts ii. a. & b. Thus Stoke's Theorem says that we should get the same answers for all three parts.

6.8 Differential k-forms

Definition 44 *For $1 \leq i_1, i_2, \ldots, i_k \leq n$, we define functions on k-tuples of vectors*

$$p^{i_1} \wedge p^{i_2} \wedge \cdots \wedge p^{i_k} \colon (\mathbb{R}^n)^k \to \mathbb{R}$$

by

$$p^{i_1} \wedge p^{i_2} \wedge \cdots \wedge p^{i_k}(\mathbf{t}_1, \ldots, \mathbf{t}_k) = \det \begin{pmatrix} t_1^{i_1} & \cdots & t_k^{i_1} \\ \vdots & \ddots & \vdots \\ t_1^{i_k} & \cdots & t_k^{i_k} \end{pmatrix}$$

for vectors $\mathbf{t}_i \in \mathbb{R}^n$, $1 \leq i \leq k$.

Examples on $\mathbb{R}^4$,

$p^1 \wedge p^3 : (\mathbb{R}^4)^2 \to \mathbb{R}$; and a particular example is

$$p^1 \wedge p^3 \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 8 \\ 0 & -4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 8 - 6 = 2.$$

$p^2 \wedge p^1 \wedge p^4 : (\mathbb{R}^4)^3 \to \mathbb{R}$; and a particular example is

$$p^2 \wedge p^1 \wedge p^4 \begin{pmatrix} 1 & 3 & 1 \\ 2 & -2 & 4 \\ 3 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{vmatrix} 2 & -2 & 4 \\ 1 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 14 + 8 = 22.$$

$p^3 : \mathbb{R}^4 \to \mathbb{R}$; and a particular example is

$$p^3 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 3.$$

Proposition 45 *(a) If two terms in $p^{i_1} \wedge \cdots \wedge p^{i_k}$ are interchanged, then the function changes sign, e.g. $p^4 \wedge p^2 \wedge p^1 = -p^1 \wedge p^2 \wedge p^4$.*

(b) If two terms in $p^{i_1} \wedge \cdots \wedge p^{i_k}$ are equal, then the function is zero, e.g. $p^1 \wedge p^3 \wedge p^1 = 0$.

Proof These results are immediate from the properties of determinants. ■

Definition 46 *A function $f : (\mathbb{R}^n)^k \to \mathbb{R}$ is said to be **multilinear** if it is linear in each component.*

For example,

$$f(s\mathbf{t}_1 + u\mathbf{t}_1', \mathbf{t}_2, \mathbf{t}_3, ..., \mathbf{t}_k) = sf(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, ..., \mathbf{t}_k) + uf(\mathbf{t}_1', \mathbf{t}_2, \mathbf{t}_3, ..., \mathbf{t}_k)$$

is the statement that f is linear in the first component.

Definition 47 *A function $f : (\mathbb{R}^n)^k \to \mathbb{R}$ is said to be **alternating** if $f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, ..., \mathbf{t}_k)$ changes sign if two of the vectors $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, ..., \mathbf{t}_k$ are interchanged,*

For example $f(\mathbf{t}_2, \mathbf{t}_1, \mathbf{t}_3) = -f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$.

Proposition 48 *The functions $p^{i_1} \wedge \cdots \wedge p^{i_k} : (\mathbb{R}^n)^k \to \mathbb{R}$ are multilinear and alternating.*

Proof This is immediate from the basic properties of determinants. ■

Note that for just two terms we talked of $p^i \wedge p^j$ being anti-symmetric instead of alternating, but they mean the same thing.

Definition 49 *The set of all alternating multilinear maps $(\mathbb{R}^n)^k \to \mathbb{R}$ is denoted $\Lambda^k(\mathbb{R}^n, \mathbb{R})$. Thus $\Lambda^1(\mathbb{R}^n, \mathbb{R}) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$.*

By convention, $\Lambda^0(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}$.

Theorem 50 *The set of all alternating multilinear maps $(\mathbb{R}^n)^k \to \mathbb{R}$, $\Lambda^k(\mathbb{R}^n, \mathbb{R})$, is a vector space with a basis given by*

$$\{\, p^{i_1} \wedge \cdots \wedge p^{i_k} \mid 1 \le i_1 < i_2 < \cdots < i_k \le n \,\}.$$

Proof Not given. ■

Corollary 51 *The dimension of the vector space $\Lambda^k(\mathbb{R}^n, \mathbb{R})$ is $\binom{n}{k}$. In particular, if $k > n$, then $\Lambda^k(\mathbb{R}^n, \mathbb{R}) = 0$.*

Proof This follows from Theorem 50 since the dimension of a vector space is given by the number of vectors in a basis. ∎

Definition 52 *Given an open set $U \subset \mathbb{R}^n$, a **(differential) k-form** on U is a function*

$$\boldsymbol{\omega} : U \to \Lambda^k(\mathbb{R}^n, \mathbb{R}).$$

As for 1-forms (Definition 2) we usually write $\boldsymbol{\omega}_{\mathbf{a}}$ in place of $\boldsymbol{\omega}(\mathbf{a})$.

Definition 53 *We write $dx^{i_1} \wedge \cdots \wedge dx^{i_k} : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n, \mathbb{R})$ for the constant k-form given by*

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}(\mathbf{x}) = p^{i_1} \wedge \cdots \wedge p^{i_k},$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Proposition 54 *(a) If two terms in $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ are interchanged, then the function changes sign, e.g. $dx^4 \wedge dx^2 \wedge dx^1 = -dx^1 \wedge dx^2 \wedge dx^4$.*

(b) If two terms in $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ are equal, then the function is zero, e.g. $dx^1 \wedge dx^3 \wedge dx^1 = 0$.

Proof This is immediate from Proposition 45. ∎

Remarks. (a) It is a consequence of Theorem 50 that each k-form $\boldsymbol{\omega} : U \to \Lambda^k(\mathbb{R}^n, \mathbb{R})$, U open in $\mathbb{R}^n$, can be written

$$\boldsymbol{\omega} = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\ldots,i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

for some functions $\omega_{i_1,\ldots,i_k} : U \subseteq \mathbb{R}^n \to \mathbb{R}$.

(b) We say that the k-form is of **class $\mathcal{C}^q$** if all these component functions, $\omega_{i_1,\ldots,i_k}$, are of class $\mathcal{C}^q$.

(c) By Corollary 51, for $k > n$, the only k-form on $U \subset \mathbb{R}^n$ is the zero form.

Definition 55 *The **wedge product** or **exterior product** of a k-form $\boldsymbol{\alpha}$ and an l-form $\boldsymbol{\beta}$ is a $(k+l)$-form $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$. To define this we put*

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_l}) = dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l},$$

and extend linearly.

Example If $\boldsymbol{\omega} = xz\,dx \wedge dy + (\sin y)\,dx \wedge dz + e^x\,dy \wedge dz$ and $\boldsymbol{\alpha} = x\,dx + z\,dy + xy\,dz$ then

$$\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\omega} &= (xz\,dx \wedge dy + (\sin y)\,dx \wedge dz + e^x\,dy \wedge dz) \wedge (x\,dx + z\,dy + xy\,dz) \\
&= xzx\,dx \wedge dy \wedge dx + xz^2 dx \wedge dy \wedge dy + x^2yz\,dx \wedge dy \wedge dz \\
&\qquad + x(\sin y)\,dx \wedge dz \wedge dx + (\sin y)z\,dx \wedge dz \wedge dy + xy(\sin y)\,dx \wedge dz \wedge dz \\
&\qquad\qquad + xe^x\,dy \wedge dz \wedge dx + e^x z\,dy \wedge dz \wedge dy + xe^x y\,dy \wedge dz \wedge dz,
\end{aligned}$$

having used only the linearity part of the definition. Now use the fact that terms with repeated differentials are zero. So

$$\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\omega} &= x^2yz\,dx \wedge dy \wedge dz + (\sin y)z\,dx \wedge dz \wedge dy + xe^x\,dy \wedge dz \wedge dx \\
&= \left(x^2yz - z\sin y + xe^x\right) dx \wedge dy \wedge dz,
\end{aligned}$$

having used the alternating properties of $\wedge$. ■

The motivation for the following definition is simply that it is what we did when differentiating a 1-form.

Definition 56 *Suppose that*

$$\boldsymbol{\omega} = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\ldots,i_k}\,dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

*is a k-form on an open set $U \subseteq \mathbb{R}^n$, where $\omega_{i_1,\ldots,i_k} : U \to \mathbb{R}$. Then the **exterior differential** $d\boldsymbol{\omega}$ is the $(k+1)$-form on U given by*

$$d\boldsymbol{\omega} = \sum_{1 \le i_1 < \cdots < i_k \le n} d\omega_{i_1,\ldots,i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Stress that the differential of a k-form is a $(k+1)$-form. The form $\boldsymbol{\omega}$ has to be of class $\mathcal{C}^\ell$ with some $\ell \geq 1$ for the $d\omega_{i_1,\ldots,i_k}$ to be defined. The differential $d\boldsymbol{\omega}$ will be of class $\mathcal{C}^{\ell-1}$.

Example.

$$\begin{aligned}
&d\big((x^2+y)\,dx\wedge dy + xyz\,dx\wedge dz + x^2z\,dy\wedge dz\big)\\
&\qquad = d\left(x^2+y\right)\wedge dx\wedge dy + d\,(xyz)\wedge dx\wedge dz + d\left(x^2z\right)\wedge dy\wedge dz\\
&\qquad = (2xdx+dy)\wedge dx\wedge dy + (yzdx+xzdy+xydz)\wedge dx\wedge dz\\
&\qquad\qquad + \left(2xzdx + x^2dz\right)\wedge dy\wedge dz\\
&\qquad = xzdy\wedge dx\wedge dz + 2xzdx\wedge dy\wedge dz\\
&\qquad = -xzdx\wedge dy\wedge dz + 2xzdx\wedge dy\wedge dz\\
&\qquad = xz\,dx\wedge dy\wedge dz.
\end{aligned}$$

■

Definition 57 *(a) A k-form $\boldsymbol{\omega}$ is* ***exact*** *when $\boldsymbol{\omega} = d\boldsymbol{\alpha}$ for a $(k-1)$-form $\boldsymbol{\alpha}$.*

(b) A k-form $\boldsymbol{\omega}$ is ***closed*** *when $d\boldsymbol{\omega} = 0$.*

In Corollary 40 it was shown that if $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, i.e. a 0-form, was $\mathcal{C}^2$ then $d^2 f = 0$. This is, in fact, true for forms of all orders.

Proposition 58 *Given a k-form $\boldsymbol{\omega}$ of class $\mathcal{C}^2$ then $d^2\boldsymbol{\omega} = d(d\boldsymbol{\omega}) = 0$.*

Proof From the definition of the exterior differential

$$d\boldsymbol{\omega} = \sum_{1\leq i_1<\cdots<i_k\leq n} d\omega_{i_1,\ldots,i_k}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Look at just one term in the sum and write $f = \omega_{i_1,\ldots,i_k}$, so the term is

$$df\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k} = \sum_{i=1}^n \frac{\partial f}{\partial x^i}dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Then $d(d\boldsymbol{\omega})$ is a sum of

$$\begin{aligned} d\left(df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) &= \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{aligned}$$

The point here is that the addition of the $\wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ makes no difference to the argument seen before. Pairing (i, j) and (j, i) pairs we get this sum equals

$$\sum_{1 \leq i < j \leq n} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j}\right) dx^j \wedge dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = 0$$

since $f = \omega_{i_1,\ldots,i_k}$ is a $\mathcal{C}^2$ function. ■

Hence, if $\boldsymbol{\omega} = d\boldsymbol{\alpha}$ for some form $\boldsymbol{\alpha}$ then $d\boldsymbol{\omega} = \mathbf{0}$. In words:

Corollary 59 *If a k-form of class C^1 is exact then it is closed.*

Remark. The converse, if $d\boldsymbol{\omega} = \mathbf{0}$ then $\boldsymbol{\omega} = d\boldsymbol{\alpha}$ for some form $\boldsymbol{\alpha}$, is only partially true. It is true 'locally': given a closed k-form $\boldsymbol{\omega}$ on an open set $U \subseteq \mathbb{R}^n$, any point $\mathbf{x} \in U$ has a neighbourhood on which there exists a $(k-1)$-form $\boldsymbol{\alpha}$ with $\boldsymbol{\omega} = d\boldsymbol{\alpha}$. It is also true if the forms defined on the whole of $\mathbb{R}^n$. The conditions under which the converse holds are the subject of Poincaré's Lemma.

6.9 Integration of a k-form over a manifold

Without defining the terms we will state a version of the general Stokes' Theorem as an advert for the type of result you might find in MATH31061 *Differentiable Manifolds.*

Theorem 60 *Let $M \subseteq \mathbb{R}^n$ be a compact orientable regular k-surface with boundary ∂M of class $\mathcal{C}^2$ and let ω be a $\mathcal{C}^1$ $(k-1)$-form on an open neighbourhood U of M. Then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

Appendix to Section 6

Motivation of definition of a 1-form and its integral.

If $f : [a,b] \to \mathbb{R}$ is integrable then we (presumably) know the definition of

$$\int_a^b f(x)\,dx.$$

We can look upon this as the integral over the curve $[a,b]$ within $\mathbb{R}$. But a curve is simply the image of a continuous map of an interval (which in this case could be the identity map). Let ϕ be differentiable map, say from $[0,1]$ to $[a,b]$, with $\phi(0) = a$ and $\phi(1) = b$. Then, by the well-known change of variables,

$$\int_a^b f(x)\,dx = \int_0^1 f(\phi(t))\,\phi'(t)\,dt.$$

Let us attempt to generalise this to paths in $\mathbb{R}^n$. A path $\boldsymbol{\gamma}$ in $\mathbb{R}^n$ will be the image of some differentiable map $\mathbf{g} : [0,1] \to \mathbb{R}^n$, so $\boldsymbol{\gamma} = \operatorname{Im}\mathbf{g}$. Assume that f is a scalar-valued function defined on the points of $\boldsymbol{\gamma}$; perhaps $f : U \to \mathbb{R}$, for some open set $U \subseteq \mathbb{R}^n$ with $\boldsymbol{\gamma} \subseteq U$. What meaning could we give to

$$\int_0^1 f(\mathbf{g}(t))\,\mathbf{g}'(t)\,dt\ ?$$

Without some definition of integrating vectors this has no meaning for $\mathbf{g}'(t)$ is the tangent **vector** to the curve $\boldsymbol{\gamma}$ at point $\mathbf{g}(t)$. Instead we introduce a linear function $\boldsymbol{\omega}$ from vectors to $\mathbb{R}$ and consider

$$\int_0^1 f(\mathbf{g}(t))\,\boldsymbol{\omega}(\mathbf{g}'(t))\,dt.$$

Except this is not quite correct. The vector $\mathbf{g}'(t)$ is a vector with a point of application, it may be that at different t_1 and t_2 the vectors $\mathbf{g}'(t_1)$ and $\mathbf{g}'(t_2)$ have the same direction and magnitude but a different point of application ($\mathbf{g}(t_1)$ and $\mathbf{g}(t_2)$ respectively). Thus we want the possibility that, at $\mathbf{g}'(t_1)$ and $\mathbf{g}'(t_2)$, $\boldsymbol{\omega}$ can be a different function, in which case we assume $\boldsymbol{\omega}$ also depends on the point of application, written as $\boldsymbol{\omega}_{\mathbf{g}(t)}$. Now $\boldsymbol{\omega}$ is a function from U such that for every $\mathbf{a} \in U$ we have a linear function $\boldsymbol{\omega}_{\mathbf{a}}$ from vectors to $\mathbb{R}$. This has led us to a 1-form.

Finally when you look at the definition of the integral of a 1-form you find no mention of a function f. But if $\boldsymbol{\omega}$ is a 1-form then $f\boldsymbol{\omega}$ is a 1-form for any $f : U \to \mathbb{R}$. Thus we can 'absorb' the f into the 1-form and consider

$$\int_0^1 \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t))\, dt$$

to be the definition of the integral of $\boldsymbol{\omega}$ over $\boldsymbol{\gamma}$.

Projections are linearly independent.

Let $p^i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq n$ be the projection functions on $\mathbb{R}^n$. Then, by definition, $p^i(\mathbf{e}_j) = \delta_{i,j}$. Assume there exist $c_i \in \mathbb{R} : \sum_{i=1}^n c_i p^i = 0$, the zero function. Then, for each $1 \leq j \leq n$, we have

$$0 = 0(\mathbf{e}_j) = \sum_{i=1}^n c_i p^i(\mathbf{e}_j) = \sum_{i=1}^n c_i \delta_{i,j} = c_j.$$

Thus, for all $1 \leq j \leq n$ we have $c_j = 0$. Hence the p^j, $1 \leq j \leq n$, are linearly independent.

Second derivatives need not be equal

Example 61 *Define*

$$f(\mathbf{x}) = \frac{xy\left(x^2 - y^2\right)}{x^2 + y^2} \text{ if } \mathbf{x} \neq \mathbf{0} \quad \text{with} \quad f(\mathbf{0}) = 0.$$

Then

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{0}) = -1 \quad \text{while} \quad \frac{\partial^2 f}{\partial y \partial x}(\mathbf{0}) = 1.$$

Verification Assume $\mathbf{x} \neq \mathbf{0}$ and multiply up so $\left(x^2 + y^2\right) f(\mathbf{x}) = xy\left(x^2 - y^2\right)$. Then

$$2yf(\mathbf{x}) + \left(x^2 + y^2\right) \frac{\partial f(\mathbf{x})}{\partial y} = x\left(x^2 - y^2\right) - 2xy^2 = x^3 - 3xy^2.$$

Put $y = 0$ (which with $\mathbf{x} \neq \mathbf{0}$ implies $x \neq 0$) to get

$$\left.\frac{\partial f(\mathbf{x})}{\partial y}\right|_{\mathbf{x}=(x,0)^T} = x. \tag{22}$$

If $x = 0$ go back to the definition,

$$\frac{\partial f}{\partial y}(\mathbf{0}) = \lim_{t \to 0} \frac{f\left((0,t)^T\right) - f(\mathbf{0})}{t} = 0. \tag{23}$$

Similarly,

$$\left.\frac{\partial f(\mathbf{x})}{\partial x}\right|_{\mathbf{x}=(0,y)^T} = \begin{cases} -y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases} \tag{24}$$

Then

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{0}) = \frac{\partial}{\partial x}\frac{\partial f(\mathbf{0})}{\partial y} = \lim_{h \to 0} \frac{1}{h}\left(\frac{\partial f\left((h,0)^T\right)}{\partial y} - \frac{\partial f}{\partial y}(\mathbf{0})\right) = \lim_{h \to 0} \frac{h}{h} = 1,$$

by (22) and (23) . Also,

$$\frac{\partial^2 f}{\partial y \partial x}(\mathbf{0}) = \frac{\partial}{\partial y}\frac{\partial f(\mathbf{0})}{\partial x} = \lim_{t \to 0} \frac{1}{t}\left(\frac{\partial f\left((0,t)^T\right)}{\partial x} - \frac{\partial f}{\partial x}(\mathbf{0})\right) = \lim_{t \to 0} \frac{-t}{t} = -1,$$

by (24). So we have an example of where the order of the derivatives is important. ■

Theorem 15

In the proof of Theorem 15 we have used the result that

$$\frac{d}{ds} f(\mathbf{a} + s\mathbf{e}_j + t\mathbf{e}_i) = \frac{\partial f}{\partial x^j}(\mathbf{a} + s\mathbf{e}_j + t\mathbf{e}_i).$$

The derivative on the left hand side is the one known since before University, the one on the right is the partial, i.e. directional, derivative defined in this course.

Let $\mathbf{y} = \mathbf{a} + t\mathbf{e}_i$ and $\mathbf{w} = \mathbf{y} + s\mathbf{e}_j$then

$$\begin{aligned} \frac{d}{ds} f(\mathbf{y} + s\mathbf{e}_j) &= \lim_{h \to 0} \frac{f(\mathbf{y} + (s+h)\mathbf{e}_j) - f(\mathbf{y} + s\mathbf{e}_j)}{h} \\ &= \lim_{h \to 0} \frac{f(\mathbf{w} + h\mathbf{e}_j) - f(\mathbf{w})}{h} = \frac{\partial f}{\partial x^j}(\mathbf{w}). \end{aligned}$$

2-forms in $\mathbb{R}^2$ and their integrals.

All 2-forms on $\mathbb{R}^2$ are of the form $\boldsymbol{\omega} = f\,dx \wedge dy$ for some $f : \mathbb{R}^2 \to \mathbb{R}$. Let $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ be Fréchet differentiable on $D' \subseteq \mathbb{R}^2$. Let $D = \mathbf{g}(D')$. Then, by definition of integration,

$$\begin{aligned}
\int_D \boldsymbol{\omega} &= \iint\limits_{D'} \boldsymbol{\omega}_{\mathbf{g}(x,y)}\left(\frac{\partial \mathbf{g}\,(x,y)}{\partial x}, \frac{\partial \mathbf{g}\,(x,y)}{\partial y}\right) dxdy \\
&= \iint\limits_{D'} f(\mathbf{g}\,(x,y))\, p^1 \wedge p^2 \left(\left(\begin{array}{c} g_x^1 \\ g_x^2 \end{array}\right), \left(\begin{array}{c} g_y^1 \\ g_y^2 \end{array}\right)\right) dxdy \\
&= \iint\limits_{D'} f(\mathbf{g}(\mathbf{x}))\, |J\mathbf{g}(\mathbf{x})|\, d\mathbf{x}. \qquad (25)
\end{aligned}$$

If we choose $\mathbf{g}$ to be the identity, so $D = D'$, we get

$$\int_D \boldsymbol{\omega} = \iint\limits_{D} f(\mathbf{u})\, d\mathbf{u},$$

i.e. the definition for 2-forms reduces to the double integrals we already know. But further, we can combine this reinterpretation of integration of a 2-form with (25) to get

$$\iint\limits_{D'} f(\mathbf{g}(\mathbf{x}))\, |J\mathbf{g}(\mathbf{x})|\, d\mathbf{x} = \int_D \boldsymbol{\omega} = \iint\limits_{D} f(\mathbf{u})\, d\mathbf{u}. \qquad (26)$$

This is the well-known formula for change of variables in an integral in $\mathbb{R}^2$. We normally talk of $|J\mathbf{g}(\mathbf{x})|$ being the Jacobian of $\mathbf{g}$. This is why I have been careful throughout the course, referring to $J\mathbf{g}(\mathbf{x})$ as the Jacobian *matrix*.

Motivation for the definition of a 2-form and its integral.

Let $S \subseteq \mathbb{R}^n$ be a surface given parametrically by a Fréchet differentiable map $\mathbf{g} : D \to \mathbb{R}^n$ for some $D \subseteq \mathbb{R}^2$. So $S = \operatorname{Im} \mathbf{g}$ and $J\mathbf{g}(s,t)$ is of rank 2 for all $(s,t)^T \in D$.

In place of functions of the vector $\mathbf{g}'(t)$ which appears in the line integral we might consider real valued functions of $d\mathbf{g}$. The Fréchet derivative $d\mathbf{g}$ can be considered as a linear map or, as $J\mathbf{g}$, a matrix. For this discussion think of $d\mathbf{g}$ as the set of its two columns, the directional derivatives $d_s\mathbf{g}$ and $d_t\mathbf{g}$. For each $\mathbf{p} \in S$, look at real valued functions $\boldsymbol{\omega}_{\mathbf{p}}$ of *pairs* $(\mathbf{v}, \mathbf{w})$ of vectors $\mathbf{v}$ and $\mathbf{w}$ with point of application $\mathbf{p} \in S$ with the hope that

$$\int\!\!\int_D \boldsymbol{\omega}_{\mathbf{g}(s,t)}(d_s\mathbf{g}, d_t\mathbf{g})\, dsdt \tag{27}$$

makes sense.

If (27) represents the integral over S it should not depend on the parametrization. So if $\mathbf{h} : D' \to \mathbb{R}^n$ also represents S in that $S = \operatorname{Im} \mathbf{h}$, the integral

$$\int\!\!\int_{D'} \boldsymbol{\omega}_{\mathbf{h}(u,v)}(d_u\mathbf{h}, d_v\mathbf{h})\, dudv$$

should have the same value as (27). Assume there exists an invertible differentiable function $\mathbf{k} : D' \to D$ such that $\mathbf{h} = \mathbf{g} \circ \mathbf{k}$. Since $\boldsymbol{\omega}_{\mathbf{g}(s,t)}(d_s\mathbf{g}, d_t\mathbf{g})$ is a real valued function of $(s,t)^T \in \mathbb{R}^2$ we can apply (25) to say

$$\int\!\!\int_D \boldsymbol{\omega}_{\mathbf{g}(s,t)}\left(d_s\mathbf{g}, d_t\mathbf{g}\right) dsdt = \int\!\!\int_{D'} \boldsymbol{\omega}_{\mathbf{g}(\mathbf{k}(u,v))}\left(d_s\mathbf{g}, d_t\mathbf{g}\right) \left|J\mathbf{k}(\mathbf{u})\right| dudv. \tag{28}$$

By the Chain Rule

$$J\mathbf{h}(\mathbf{u}) = J\mathbf{g}(\mathbf{k}(\mathbf{u}))\, J\mathbf{k}(\mathbf{u}) = J\mathbf{g}(\mathbf{s})\, J\mathbf{k}(\mathbf{u}) = (d_s\mathbf{g}, d_t\mathbf{g}) \begin{pmatrix} d_u k^1 & d_v k^1 \\ d_u k^2 & d_v k^2 \end{pmatrix}.$$

To cut down notation write this as

$$(d_u\mathbf{h}, d_v\mathbf{h}) = \left(\alpha d_s\mathbf{g} + \beta d_t\mathbf{g},\ \gamma d_s\mathbf{g} + \delta d_t\mathbf{g}\right).$$

Forgetting temporarily the subscript

$$\begin{aligned}
\boldsymbol{\omega}(d_u\mathbf{h}, d_v\mathbf{h}) &= \boldsymbol{\omega}(\alpha d_s\mathbf{g} + \beta d_t\mathbf{g},\, \gamma d_s\mathbf{g} + \delta d_t\mathbf{g}) \\
&= \alpha\gamma\boldsymbol{\omega}(d_s\mathbf{g}, d_s\mathbf{g}) + \alpha\delta\boldsymbol{\omega}(d_s\mathbf{g},\, d_t\mathbf{g}) \qquad (29) \\
&\qquad + \beta\gamma\boldsymbol{\omega}(d_t\mathbf{g},\, d_s\mathbf{g}) + \beta\delta\boldsymbol{\omega}(d_t\mathbf{g},\, d_t\mathbf{g}) \qquad (30)
\end{aligned}$$

assuming $\boldsymbol{\omega}_{\mathbf{p}}$ is linear in both arguments (bilinear). What further properties should we demand of $\boldsymbol{\omega}$?

In some sense $\boldsymbol{\omega}_{\mathbf{p}}(\mathbf{u}, \mathbf{v})$ should represent the area between $\mathbf{u}$ and $\mathbf{v}$. In particular if $\mathbf{u}$ and $\mathbf{v}$ are parallel then $\boldsymbol{\omega}_{\mathbf{p}}(\mathbf{u}, \mathbf{v}) = 0$. If $\mathbf{u}$ and $\mathbf{v}$ are interchanged then $\boldsymbol{\omega}_{\mathbf{p}}(\mathbf{u}, \mathbf{v})$ should change sign, i.e. $\boldsymbol{\omega}_{\mathbf{p}}(\mathbf{v}, \mathbf{u}) = -\boldsymbol{\omega}_{\mathbf{p}}(\mathbf{u}, \mathbf{v})$. In (29) this give

$$\boldsymbol{\omega}(d_u\mathbf{h}, d_v\mathbf{h}) = (\alpha\delta - \beta\gamma)\,\boldsymbol{\omega}(d_s\mathbf{g},\, d_t\mathbf{g}) = |J\left(\mathbf{k}\left(\mathbf{u}\right)\right)|\,\boldsymbol{\omega}(d_s\mathbf{g},\, d_t\mathbf{g})\,,$$

having recalled the definitions of $\alpha, \beta, \ldots$ Returning to (28) ,

$$\begin{aligned}
\int\!\!\int_D \boldsymbol{\omega}_{\mathbf{g}(s,t)}(d_s\mathbf{g}, d_t\mathbf{g})\, dsdt &= \int\!\!\int_{D'} \boldsymbol{\omega}_{\mathbf{h}(u,v)}(d_s\mathbf{g}, d_t\mathbf{g})\, |J\mathbf{k}\left(\mathbf{u}\right)|\, dudv \\
&= \int\!\!\int_{D'} \boldsymbol{\omega}_{\mathbf{h}(u,v)}(d_u\mathbf{h}, d_v\mathbf{h})\, dudv.
\end{aligned}$$

Thus under the properties of $\boldsymbol{\omega}(\mathbf{u}, \mathbf{v}) = -\boldsymbol{\omega}(\mathbf{v}, \mathbf{u})$ and linearity in both variables we have that the integral over S does not depend on the parametrization.

The constant 2-forms $dx^i \wedge dx^j$, $1 \leq i < j \leq n$ are linearly independent.

Proof Assume there exist $\omega_{i,j} : U \to \mathbb{R}$ such that

$$\sum_{1\leq i<j\leq n} \omega_{i,j} dx^i \wedge dx^j = 0,$$

an equality of 2-forms. Then, for any $\mathbf{a} \in U$,

$$\sum_{1\leq i<j\leq n} \omega_{i,j}\left(\mathbf{a}\right) dx^i \wedge dx^j\left(\mathbf{a}\right) = \sum_{1\leq i<j\leq n} \omega_{i,j}\left(\mathbf{a}\right) p^i \wedge p^j = 0,$$

the final equality as functions $(\mathbb{R}^n)^2 \to \mathbb{R}$. Note now that, with $1 \leq k < \ell \leq n$,

$$p^i \wedge p^j \left((\mathbf{e}_k, \mathbf{e}_\ell) \right) = \begin{cases} 1 & \text{if } (i,j) = (k,\ell) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$0 = \sum_{1 \leq i < j \leq n} \omega_{i,j} \left(\mathbf{a} \right) p^i \wedge p^j \left((\mathbf{e}_k, \mathbf{e}_\ell) \right) = \omega_{k\ell} \left(\mathbf{a} \right).$$

Thus $\omega_{k\ell} \left(\mathbf{a} \right) = 0$ for any $\mathbf{a} \in U$, hence $\omega_{k\ell} = 0$.

True for any $1 \leq k < \ell \leq n$ implies the $dx^i \wedge dx^j$ are linearly independent.

■

Motivation for Differential of a 1-form

Let $\boldsymbol{\omega} : U \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ be a 1-form. Ideally a differential of $\boldsymbol{\omega}$ at $\mathbf{p} \in U$ would measure the 'change' in $\boldsymbol{\omega}_\mathbf{p}$ as $\mathbf{p}$ varies. For a scalar-valued function f of several variables we could measure its rate of change at a point $\mathbf{p}$ in direction $\mathbf{v}$ by $\nabla f(\mathbf{p}) \bullet \mathbf{v}$. In the present case $\boldsymbol{\omega}_\mathbf{p}$ is **not** a scalar-valued function of $\mathbf{p}$. It is, though, if evaluated *on* a vector.

If the differential is to be a 2-form assume we are given two vectors V_1 and V_2, written upper case so we think of them as constant vector fields on $\mathbb{R}^n$. Given $\mathbf{p} \in U$ consider $\boldsymbol{\omega}_\mathbf{p}(V_1)$ and $\boldsymbol{\omega}_\mathbf{p}(V_2)$ as functions of $\mathbf{p}$, so scalar-valued functions $U \to \mathbb{R}$. We look at the rate of change of $\boldsymbol{\omega}_\mathbf{p}(V_1)$ as $\mathbf{p}$ moves in the V_2 direction, and the rate of change of $\boldsymbol{\omega}_\mathbf{p}(V_2)$ as $\mathbf{p}$ moves in the V_1 direction. That is, we take the gradients w.r.t $\mathbf{p}$, $\nabla \boldsymbol{\omega}_\mathbf{p}(V_1)$ and $\nabla \boldsymbol{\omega}_\mathbf{p}(V_2)$ and calculate $\nabla \boldsymbol{\omega}_\mathbf{p}(V_1) \bullet V_2$ and $\nabla \boldsymbol{\omega}_\mathbf{p}(V_2) \bullet V_1$. Finally, to get an anti-symmetric form perhaps define the differential $d\boldsymbol{\omega}_\mathbf{p}(V_1, V_2)$ as

$$\nabla \boldsymbol{\omega}_\mathbf{p}(V_2) \bullet V_1 - \nabla \boldsymbol{\omega}_\mathbf{p}(V_1) \bullet V_2. \tag{31}$$

Theorem 62 *Let* $\boldsymbol{\omega} : U \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ *be a 1-form. Let* V_1 *and* V_2 *be constant vector fields on* $\mathbb{R}^n$*. Given* $\mathbf{p} \in U$ *let*

$$d\boldsymbol{\omega}_\mathbf{p}(V_1, V_2) = \nabla \boldsymbol{\omega}_\mathbf{p}(V_2) \bullet V_1 - \nabla \boldsymbol{\omega}_\mathbf{p}(V_1) \bullet V_2.$$

Then

$$d\boldsymbol{\omega} = \sum_{i=1}^n d\omega_i \wedge dx^i.$$

Proof Write $\boldsymbol{\omega} = \sum_{i=1}^{n} \omega_i dx^i$ where $\omega_i : U \to \mathbb{R}$ are 0-forms, and assume V_1, V_2 have coordinates v_1^i and v_2^i respectively. Then

$$\boldsymbol{\omega}_{\mathbf{p}}(V_2) = \sum_{i=1}^{n} \omega_i(\mathbf{p})\, v_2^i \quad \text{so} \quad \nabla \boldsymbol{\omega}_{\mathbf{p}}(V_2) \bullet V_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i(\mathbf{p})}{\partial x^j} v_1^j v_2^i.$$

Thus

$$\nabla \boldsymbol{\omega}_{\mathbf{p}}(V_2) \bullet V_1 - \nabla \boldsymbol{\omega}_{\mathbf{p}}(V_1) \bullet V_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i(\mathbf{p})}{\partial x^j} \left(v_1^j v_2^i - v_2^j v_1^i \right).$$

But

$$\begin{aligned} v_1^j v_2^i - v_2^j v_1^i &= p^j(V_1)\, p^i(V_2) - p^i(V_1)\, p^j(V_2) \\ &= \big(p^j \wedge p^i\big)(V_1, V_2) \\ &= dx^j \wedge dx^i\, (\mathbf{a})\, (V_1, V_2)\,, \end{aligned}$$

for any $\mathbf{a} \in U$. Choose $\mathbf{a} = \mathbf{p}$ to get

$$\nabla \boldsymbol{\omega}_{\mathbf{p}}(V_2) \bullet V_1 - \nabla \boldsymbol{\omega}_{\mathbf{p}}(V_1) \bullet V_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i(\mathbf{p})}{\partial x^j} dx^j \wedge dx^i(\mathbf{p})(V_1, V_2)\,.$$

The form on the right hand side is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \right) \wedge dx^i = \sum_{i=1}^{n} d\omega_i \wedge dx^i,$$

having used the linearity of the wedge product for the first equality, and the derivative of a 0-form (see (16)). ■

Green's Theorem

Theorem 63 ***Green's Theorem*** *Let C be a closed $\mathcal{C}^1$ curve in $\mathbb{R}^2$ orientated counterclockwise and D be the interior of C. If $P(x,y)$ and $Q(x,y)$ are both functions of class $\mathcal{C}^1$ then*

$$\int_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Proof The integrand $Pdx + Qdy$ is an example of a 1-form on $\mathbb{R}^2$ (and all 1-forms are of this form). Then

$$\begin{aligned} d(Pdx + Qdy) &= dP \wedge dx + dQ \wedge dy \\ &= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy \\ &= \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy. \end{aligned}$$

(This is, in fact, just a special case of Proposition 38.)

In the definition of integration over a surface

$$\int_S \boldsymbol{\omega} = \int\int_D \boldsymbol{\omega}_{\mathbf{g}(s,t)}\big(d_1\mathbf{g}(s,t), d_2\mathbf{g}(s,t)\big)dsdt.$$

choose the identity map $\mathbf{g}(s,t) = (s,t)^T \in \mathbb{R}^2$. Then $d_1\mathbf{g}(s,t) = (1,0)^T$ and $d_2\mathbf{g}(s,t) = (0,1)^T$. With $\boldsymbol{\omega} = (\partial Q/\partial x - \partial P/\partial y)\, dx \wedge dy$ we get

$$\begin{aligned} &\boldsymbol{\omega}_{\mathbf{g}(s,t)}\big(d_1\mathbf{g}(s,t), d_2\mathbf{g}(s,t)\big) \\ &\quad = \left(\frac{\partial Q}{\partial x}(s,t) - \frac{\partial P}{\partial y}(s,t)\right) dx \wedge dy(s,t)\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) \\ &\quad = \left(\frac{\partial Q}{\partial x}(s,t) - \frac{\partial P}{\partial y}(s,t)\right) p^1 \wedge p^2 \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) \\ &\quad = \frac{\partial Q}{\partial x}(s,t) - \frac{\partial P}{\partial y}(s,t). \end{aligned}$$

Hence result follows from Stokes' Theorem. ■

Though we have deduced Green's Theorem from Stokes' Theorem it is important to have (and many exist) independent proofs of Green's Theorem. This is because many proofs of Stokes' Theorem reduce it to an application of Green's Theorem.

Divergence of Vector Fields

Restrict to $\mathbb{R}^3$. Let $\mathbf{f} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Let $\boldsymbol{\omega} = \mathbf{f} \bullet \mathbf{n}$, where

$$\mathbf{n} = \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix}.$$

Thus

$$\omega = f^1 dy \wedge dz + f^2 dz \wedge dx + f^3 dx \wedge dy,$$

the 2 -form associated with $\mathbf{f}$. Also

$$\begin{aligned} d\boldsymbol{\omega} &= \frac{\partial f^1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial f^2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial f^3}{\partial z} dz \wedge dx \wedge dy \\ &= \left(\frac{\partial f^1}{\partial x} + \frac{\partial f^2}{\partial y} + \frac{\partial f^3}{\partial z} \right) dx \wedge dy \wedge dz. \end{aligned}$$

Definition 64 *For* $\mathbf{f}$*, a vector field on an open subset* $U \subseteq \mathbb{R}^3$*, the* ***divergence*** *of* $\mathbf{f}$ *is*

$$\text{div}\mathbf{f} = \frac{\partial f^1}{\partial x} + \frac{\partial f^2}{\partial y} + \frac{\partial f^3}{\partial z}$$

Thus $d\boldsymbol{\omega} = (\text{div}\mathbf{f})\, dx \wedge dy \wedge dz$.

In the general form of Stokes' Theorem let $S \subseteq \mathbb{R}^3$ be a three dimensional subset with a boundary ∂S, a two dimensional surface. Then

$$\int_{\partial S} \mathbf{f} \bullet \mathbf{n} = \int_S d\left(\mathbf{f} \bullet \mathbf{n}\right) = \int_S \left(\text{div}\mathbf{f}\right) dxdydz,$$

i.e.

$$\int_{\partial S} \mathbf{f} \bullet \mathbf{n} = \int_S \left(\text{div}\mathbf{f}\right) dxdydz.$$

This is known as the **Divergence Theorem**.